A simple construction of the continuum parabolic Anderson model on $\mathbf{R}^2$

January 4, 2015

Martin Hairer[1] **and Cyril Labbé**[2]

[1] University of Warwick, Email: `M.Hairer@Warwick.ac.uk`
[2] University of Warwick, Email: `C.Labbe@Warwick.ac.uk`

Abstract

We propose a simple construction of the solution to the continuum parabolic Anderson model on $\mathbf{R}^2$ which does not rely on any elaborate arguments and makes extensive use of the linearity of the equation. A logarithmic renormalisation is required to counterbalance the divergent product appearing in the equation. Furthermore, we use time-dependent weights in our spaces of distributions in order to construct the solution on the unbounded space $\mathbf{R}^2$.

1 Introduction

The goal of this note is to construct solutions to the continuous parabolic Anderson model:

$$\partial_t u = \Delta u + u \cdot \xi \;, \qquad u(0,x) = u_0(x) \;. \tag{PAM}$$

Here, u is a function of $t \geq 0$ and $x \in \mathbf{R}^2$, while ξ is a white noise on $\mathbf{R}^2$. Notice that ξ is constant in time, so this is quite different from the model studied for example in [CM94, CJK13]. The difficulty of this problem is twofold. First, the product $u \cdot \xi$ is not classically well-defined since the sum of the Hölder regularities of u and ξ is slightly below 0. Second, our space variable x lies in the unbounded space $\mathbf{R}^2$ so that one needs to incorporate weights in the Hölder spaces at stake; this causes some difficulty in obtaining the fixed point argument, since one would a priori require a larger weight for $u \cdot \xi$ than for u itself.

The first issue is handled thanks to a renormalisation procedure which, informally, consists in subtracting an infinite linear term from the original equation. The main trick that spares us from using elaborate renormalisation theories is to introduce the "stationary" solution Y of the (additive) stochastic heat equation and to solve the PDE associated to $v = ue^Y$ instead of u. This is analogous to what was done for example in [DPD02, HPP13]. The second issue is dealt with by choosing an appropriate time-increasing weight for the solution u. Roughly speaking, if ξ is weighted by the polynomial function $\mathrm{p}_a(x) = (1 + |x|)^a$ with a small, and u_s is weighted by the exponential function $e_s(x) = e^{s(1+|x|)}$, then $\int_0^t P_{t-s} * (u_s \cdot \xi)(x)\,ds$ requires a weight of order $\int_0^t p_a(x) e_s(x)\,ds$, which is smaller than $e_t(x)$. This argument already appears in [HPP13], and probably also elsewhere in the PDE literature.

The solution to the (generalised) parabolic Anderson model has already been constructed independently by Gubinelli, Imkeller and Perkowski [GIP12] and by Hairer [Hai14] in dimension 2 and, to some extent, by Hairer and Pardoux [HP14] in dimension 3. (The latter actually considers the case of dimension 1 with space-time white noise, but the case of dimension 3 with spatial noise has exactly the same scaling behaviour, so the proof given there carries through *mutatis mutandis*. The main difference is that some of the renormalisation constants that converge to finite limits in [HP14] may diverge logarithmically.) However, in all of these results the space variable is restricted to a torus, which is the constraint that we lift in this note. The construction that we propose here is very specific to (PAM) in dimension 2: in particular, as it stands, it unfortunately applies neither to the generalised parabolic Anderson model considered in [GIP12, Hai14], nor to the case of dimension 3.

Let us now present the main steps of our construction. First, we introduce a mollified noise $\xi_\epsilon := \varrho_\epsilon * \xi$, where ϱ is a compactly supported, even, smooth function on $\mathbf{R}^2$ that integrates to 1, and $\varrho_\epsilon(x) := \epsilon^{-2}\varrho(\frac{x}{\epsilon})$ for all $x \in \mathbf{R}^2$. In order to quantify the Hölder regularity of ξ, ξ_ϵ, we introduce weighted Hölder spaces of distributions, see Section 2 below for the general definitions. Informally speaking, given a weight w and an exponent α, $\mathcal{C}^\alpha_w$ consists of those elements of $\mathcal{C}^\alpha$ that grow at most as fast as w at infinity. We have the following very simple convergence result, the proof of which is given on Page 5 below.

Lemma 1.1 *For any given $a > 0$, let $p_a(x) = (1 + |x|)^a$ on $\mathbf{R}^2$ as above. For every $\epsilon, \kappa > 0$, ξ_ϵ belongs almost surely to $\mathcal{C}^{-1-\kappa}_{p_a}(\mathbf{R}^2)$. As $\epsilon \downarrow 0$, ξ_ϵ converges in probability to ξ in $\mathcal{C}^{-1-\kappa}_{p_a}$.*

From now on, a is taken arbitrarily small. Since, for any fixed $\varepsilon > 0$, the mollified noise ξ_ε is a smooth function belonging to $\mathcal{C}^\alpha_{p_a}$ for any $\alpha > 0$, the SPDE

$$\partial_t u_\epsilon = \Delta u_\epsilon + u_\epsilon(\xi_\epsilon - C_\epsilon)\;, \qquad u_\epsilon(0,x) = u_0(x)\;, \tag{PAM$_\epsilon$}$$

is well-posed, as can be seen for example by using its Feynman-Kac representation. The constant C_ϵ appearing in this equation is required in order to control the limit $\epsilon \to 0$ and will be determined later on.

Second, let G be a compactly supported, even, smooth function on $\mathbf{R}^2 \backslash \{0\}$, such that $G(x) = -\frac{\log|x|}{2\pi}$ whenever $|x| \le \frac{1}{2}$. Then, there exists a compactly supported smooth function F on $\mathbf{R}^2$ that vanishes on the ball of radius $\frac{1}{2}$ and such that, in the distributional sense, we have:

$$\Delta G(x) = \delta_0(x) + F(x)\;. \tag{1.1}$$

With these notations at hand, we introduce the process $Y_\epsilon(x) := G * \xi_\epsilon(x)$. By construction, Y_ϵ is a smooth stationary process on $\mathbf{R}^2$ that coincides with the solution of the Poisson equation driven by ξ_ϵ, up to some smooth term:

$$\Delta Y_\epsilon(x) = \xi_\epsilon(x) + F * \xi_\epsilon(x) .$$

From now on, D_{x_i} denotes the differentiation operator with respect to the variable x_i, with $i \in \{1,2\}$. More generally, for every $\ell \in \mathbf{N}^2$, we define $D_x^\ell f$ as the map obtained from f by differentiating ℓ_1 times in direction x_1 and ℓ_2 times in direction x_2. We also use the notation $\nabla f = (D_{x_1} f, D_{x_2} f)$. The following result is a consequence of Lemma 1.1 together with the smoothing effect of the convolution with G and $D_{x_i}G$.

Corollary 1.2 *For any given $\kappa \in (0,1/2)$, the sequence of processes Y_ϵ (resp. $D_{x_i} Y_\epsilon$) converges in probability as $\epsilon \to 0$ in the space $\mathcal{C}^{1-\kappa}_{\mathrm{p}_a}(\mathbf{R}^2)$ (resp. $\mathcal{C}^{-\kappa}_{\mathrm{p}_a}(\mathbf{R}^2)$) towards the process Y (resp. $D_{x_i} Y$) defined by*

$$Y := G * \xi , \qquad D_{x_i} Y := D_{x_i} G * \xi .$$

We introduce $v_\epsilon(t,x) := u_\epsilon(t,x) e^{Y_\epsilon(x)}$ for all $x \in \mathbf{R}^2$ and $t \geq 0$, and we observe that

$$\partial_t v_\epsilon = \Delta v_\epsilon + v_\epsilon (Z_\epsilon - F * \xi_\epsilon) - 2\nabla v_\epsilon \cdot \nabla Y_\epsilon , \qquad v_\epsilon(0,x) = u_0(x) e^{Y_\epsilon(x)} ,$$

where we have introduced the renormalised process

$$Z_\epsilon(x) := |\nabla Y_\epsilon(x)|^2 - C_\epsilon .$$

At this stage we fix the renormalisation constant C_ϵ to be given by

$$C_\epsilon := \mathbb{E}\big[|\nabla Y_\epsilon|^2\big] = -\frac{1}{2\pi} \log \epsilon + \mathcal{O}(1) , \tag{1.2}$$

where the part denoted by $\mathcal{O}(1)$ converges to a constant (depending on the choice of G) as $\epsilon \to 0$. The following result, which is proven on Page 8, shows that this sequence of renormalised processes also converges in an appropriate space. We refer to Nualart [Nua06] for details on Wiener chaoses.

Proposition 1.3 *For any given $\kappa \in (0,1/2)$, the collection of processes Z_ϵ converges in probability as $\epsilon \to 0$, in the space $\mathcal{C}^{-\kappa}_{\mathrm{p}_a}(\mathbf{R}^2)$, towards the generalised process Z defined as follows: for every test function η, $\langle Z, \eta \rangle$ is the random variable in the second homogeneous Wiener chaos associated to ξ represented by the $L^2(dz\, d\tilde z)$ function*

$$(z, \tilde z) \mapsto \int \sum_{i=1,2} D_{x_i} G(z - x) D_{x_i} G(\tilde z - x) \eta(x) dx .$$

We are now able to set up a fixed point argument for the process v_ϵ with controls that are uniform in ϵ. The precise statement of the theorem requires us to introduce an appropriate Banach space $\mathcal{E}^r_{\ell,T}$; at first reading, one can replace convergence in $\mathcal{E}^r_{\ell,T}$ by locally uniform convergence in $(0,T] \times \mathbf{R}^2$. For any $r > 0$, $\ell \in \mathbf{R}$ and $T > 0$, we consider the Banach space $\mathcal{E}^r_{\ell,T}$ of all continuous functions v on $(0,T] \times \mathbf{R}^2$ such that

$$\|v\|_{\ell,T} := \sup_{t \in (0,T]} \frac{\|v_t\|_{r, \mathrm{e}_{\ell+t}}}{t^{-1+\kappa}} < \infty .$$

This leads us to the main result of this article.

Theorem 1.4 *Consider an initial condition $u_0 \in \mathcal{C}^{-1+4\kappa}_{e_\ell}$ and a final time $T > 0$. For all $\ell' > \ell$, the sequence of processes v_ϵ converges in probability as $\epsilon \to 0$ in the space $\mathcal{E}^{1+2\kappa}_{\ell',T}$ to a limit v which is the unique solution of*

$$\partial_t v = \Delta v + v(Z - F * \xi) - 2\nabla v \cdot \nabla Y \;, \qquad v(0,x) = u_0(x) e^{Y(x)} \;.$$

As a consequence, u_ϵ converges in probability in $\mathcal{E}^{1-\kappa}_{\ell',T}$ towards the process $u = ve^{-Y}$.

2 Weighted Hölder spaces

In this section, we introduce the appropriate weighted spaces that will allow us to set up a fixed point argument associated to (PAM). We work in $\mathbf{R}^d$ for a general dimension $d \in \mathbf{N}$, even though we will apply these results to $d = 2$ in the next sections.

Definition 2.1 A function $w : \mathbf{R}^d \to (0,\infty)$ is a weight if there exists a positive constant $C > 0$ such that

$$C^{-1} \le \sup_{|x-y|\le 1} \frac{w(x)}{w(y)} \le C \;.$$

In this article, we will consider two families of weights indexed by $a, \ell \in \mathbf{R}$:

$$\mathrm{p}_a(x) := (1+|x|)^a \;, \qquad \mathrm{e}_\ell(x) := \exp\left(\ell(1+|x|)\right) \;.$$

Observe that the constant C can be taken uniformly for all p_a and e_ℓ, as long as a and ℓ lie in a compact domain of $\mathbf{R}^2$. We can now consider weighted versions of the usual spaces of Hölder functions $\mathcal{C}^\alpha(\mathbf{R}^d)$.

Definition 2.2 For $\alpha \in (0,1)$, $\mathcal{C}^\alpha_w(\mathbf{R}^d)$ is the space of functions $f : \mathbf{R}^d \to \mathbf{R}$ such that

$$\|f\|_{\alpha,w} := \sup_{x\in\mathbf{R}^d} \frac{|f(x)|}{w(x)} + \sup_{|x-y|\le 1} \frac{|f(x)-f(y)|}{w(x)|x-y|^\alpha} < \infty \;.$$

More generally, for every $\alpha > 1$, we define $\mathcal{C}^\alpha_w(\mathbf{R}^d)$ recursively as the space of functions f which admit first order derivatives and such that

$$\|f\|_{\alpha,w} := \sup_{x\in\mathbf{R}^d} \frac{|f(x)|}{w(x)} + \sum_{i=1}^d \|D_{x_i} f\|_{\alpha-1,w} < \infty \;.$$

We then extend this definition to negative α. To this end, we define for every $r \in \mathbf{N}$, the space $\mathcal{B}^r_1$ of all smooth functions η on $\mathbf{R}^d$, which are compactly supported in the unit ball of $\mathbf{R}^d$ and whose $\mathcal{C}^r$ norm is smaller than 1. We will use the notation η^λ_x to denote the function $y \mapsto \lambda^{-d}\eta(\frac{y-x}{\lambda})$.

Definition 2.3 For every $\alpha < 0$, we set $r := -\lfloor \alpha \rfloor$ and we define $\mathcal{C}^\alpha_w(\mathbf{R}^d)$ as the space of distributions f on $\mathbf{R}^d$ such that

$$\|f\|_{\alpha,w} := \sup_{x\in\mathbf{R}^d} \sup_{\eta\in\mathcal{B}^r_1} \sup_{\lambda\in(0,1]} \frac{|f(\eta^\lambda_x)|}{w(x)\lambda^\alpha} < \infty \;.$$

In order to deal with the regularity of random processes, it is convenient to have a characterisation of $\mathcal{C}^\alpha_w$ that only relies on a countable number of test functions. To state such a characterisation, we need some notation. For any $\psi \in \mathcal{C}^r$, we set

$$\psi^n_x(y) := 2^{\frac{nd}{2}}\psi((y_1-x_1)2^n,\dots,(y_d-x_d)2^n)\;,\quad x,y\in\mathbf{R}^d\;,\quad n\geq 0\;.$$

We also define $\Lambda_n := \{(2^{-n}k_i)_{i=1\dots d} : k_i\in\mathbf{Z}\}$.

Proposition 2.4 *Let $\alpha < 0$ and $r > |\alpha|$. There exists a finite set Ψ of compactly supported functions in $\mathcal{C}^r$, as well as a compactly supported function $\varphi\in\mathcal{C}^r$ such that $\{\varphi^0_x, x\in\Lambda_0\}\cup\{\psi^n_x, n\geq 0, x\in\Lambda_n,\psi\in\Psi\}$ forms an orthonormal basis of $\mathbf{R}^d$, and such that for any distribution ξ on $\mathbf{R}^d$, the following equivalence holds: $\xi\in\mathcal{C}^\alpha_w$ if and only if ξ belongs to the dual of $\mathcal{C}^r$ and*

$$\sup_{n\geq 0}\sup_{\psi\in\Psi}\sup_{x\in\Lambda_n}\frac{|\langle\xi,\psi^n_x\rangle|}{w(x)2^{-\frac{nd}{2}-n\alpha}}+\sup_{x\in\Lambda_0}\frac{|\langle\xi,\varphi^0_x\rangle|}{w(x)}<\infty\;. \tag{2.1}$$

Proof. This result is rather standard and is obtained by a wavelet analysis, see [Mey92, Dau88] or [Hai14, Prop. 3.20]. In these references, the spaces are not weighted, but since all the arguments needed for the proof are local, it suffices to use the fact that $\frac{w(y)}{w(x)}$ is bounded from above and below uniformly over all x,y such that $|x-y|\leq 1$ to obtain our statement. □

Remark 2.5 If ξ is a linear transformation acting on the linear span of the functions φ^0_x, ψ^n_x such that (2.1) is finite, then ξ can be extended uniquely to an element of $\mathcal{C}^\alpha_w$.

We are now in position to characterise the regularity of the noise.

Proof of Lemma 1.1. We work in dimension $d=2$. By Proposition 2.4, it suffices to show that almost surely

$$\sup_{n\geq 0}\sup_{\psi\in\Psi}\sup_{x\in\Lambda_n}\frac{|\langle\xi,\psi^n_x\rangle|}{2^{-n(1+\alpha)}\mathrm{p}_a(x)}\lesssim 1\;,\qquad\sup_{x\in\Lambda_0}\frac{|\langle\xi,\varphi^0_x\rangle|}{\mathrm{p}_a(x)}\lesssim 1\;.$$

We restrict to the first bound, since the second is simpler. For any integer $p\geq 1$, we write

$$\mathbb{E}\Big[\sup_{n\geq 0}\sup_{\psi\in\Psi}\sup_{x\in\Lambda_n}\Big(\frac{|\langle\xi,\psi^n_x\rangle|}{2^{-n(\alpha+1)}\mathrm{p}_a(x)}\Big)^{2p}\Big]\lesssim\sum_{n\geq 0}\sum_{\psi\in\Psi}\sum_{x\in\Lambda_n}\frac{2^{2np(\alpha+1)}}{\mathrm{p}_a(x)^{2p}}(\mathbb{E}\langle\xi,\psi^n_x\rangle^2)^p$$

where we used the equivalence of moments of Gaussian random variables. Recall that the L^2 norm of ψ^n_x is 1, that the restriction of Λ_n to the unit ball of $\mathbf{R}^2$ has at most of the order of 2^{2n} elements and that Ψ is a finite set. Taking p large enough, we deduce that the triple sum converges, so that ξ admits a modification that almost surely belongs to $\mathcal{C}^\alpha_{\mathrm{p}_a}$. We now turn to $\|\xi_\epsilon-\xi\|_{\alpha,\mathrm{p}_a}$: the computation is very similar, the only difference rests on the term

$$\mathbb{E}\langle\xi-\xi_\epsilon,\psi^{n,\mathfrak{s}}_x\rangle^2=\|\psi^n_0-\varrho_\epsilon*\psi^n_0\|^2_{L^2}\lesssim 1\wedge(\epsilon^2 2^{2n})\;. \tag{2.2}$$

For p large enough and $\alpha < -1$, we obtain

$$\mathbb{E}\left[\sup_{n\geq 0}\sup_{\psi\in\Psi}\sup_{x\in\Lambda_n^{\mathfrak{s}}}\left(\frac{|\langle \xi-\xi_\epsilon,\psi_x^n\rangle|}{2^{-n(\alpha+1)}\mathrm{p}_a(x)}\right)^{2p}\right] \lesssim \sum_{x\in\mathbf{Z}^2}\sum_{n\geq 0}\frac{2^{2n+2np(\alpha+1)}}{\mathrm{p}_a(x)^{2p}}(1\wedge\epsilon^{2p}2^{2np})$$
$$\lesssim (\epsilon^{2p}|\log_2\epsilon|)\vee\epsilon^{-2\left(1+p(\alpha+1)\right)}\;.$$

Consequently, for p large enough $\mathbb{E}\|\xi_\epsilon-\xi\|_{\alpha,\mathrm{p}_a}\to 0$ as $\epsilon\downarrow 0$. □

Let w_f and w_g be two weights on $\mathbf{R}^d$. We have the following elementary extension of the classical theorem [BCD11, Thm 2.52].

Theorem 2.6 *Let $f\in\mathcal{C}^\alpha_{w_f}$ and $g\in\mathcal{C}^\beta_{w_g}$ where $\alpha<0$ and $\beta>0$ with $\alpha+\beta>0$. Then there exists a continuous bilinear map $(f,g)\mapsto f\cdot g$ from $\mathcal{C}^\alpha_{w_f}\times\mathcal{C}^\beta_{w_g}$ into $\mathcal{C}^\alpha_{w_f w_g}$ that extends the classical multiplication of smooth functions.*

Proof. Let χ be a compactly supported, smooth function on $\mathbf{R}^d$ such that $\sum_{k\in\mathbf{Z}^d}\chi(x-k)=1$ for all $x\in\mathbf{R}^d$. For simplicity, we set $\chi_k(\cdot):=\chi(\cdot-k)$. Writing $\|\cdot\|_\alpha$ for the α-Hölder norm without weight (i.e. with weight 1), observe that $h\in\mathcal{C}^\alpha_w$ if and only if $\|h\chi_k\|_\alpha\lesssim w(k)$ hold uniformly over all $k\in\mathbf{Z}^d$, and $\|h\|_{\alpha,w}$ is equivalent to the smallest possible bound. From [BCD11, Thm 2.52], we know that $f\chi_k\cdot g\chi_\ell$ is well-defined for all $k,\ell\in\mathbf{Z}^d$, and that the bound $\|f\chi_k\cdot g\chi_\ell\|_\alpha\lesssim\|f\chi_k\|_\alpha\|g\chi_\ell\|_\beta$ holds. Consequently, we get

$$\|f\chi_k\cdot g\chi_\ell\|_\alpha\lesssim w_f(k)w_g(\ell)\|f\|_{\alpha,w_f}\|g\|_{\beta,w_g}\;,$$

uniformly over all $k,\ell\in\mathbf{Z}^d$. Since the number of non-zero terms among $\{\langle f\chi_k\cdot g\chi_\ell,\eta_x\rangle, k,\ell\in\mathbf{Z}^d\}$ is uniformly bounded over all $\eta\in\mathcal{B}_1^r$, all $x\in\mathbf{R}^d$ and all f,g as in the statement, we deduce that $f\cdot g:=\sum_{k,\ell\in\mathbf{Z}^d}f\chi_k\cdot g\chi_\ell$ is well-defined and that $\|f\cdot g\|_{\alpha,w_f w_g}\lesssim\|f\|_{\alpha,w_f}\|g\|_{\beta,w_g}$ holds. Finally, the multiplication of [BCD11, Thm 2.52] extends the classical multiplication of smooth functions, therefore, from our construction, it is plain that this property still holds in our case. □

Let now $P_t(x):=(2\pi t)^{-\frac{d}{2}}e^{-|x|^2/4t}$ be the heat kernel in dimension d. We write $P_t * f$ for the spatial convolution of P_t with a function/distribution f on $\mathbf{R}^d$. We have the following regularisation property which is a slight variant of well-known facts.

Lemma 2.7 *For every $\beta\geq\alpha$ and every $f\in\mathcal{C}^\alpha_{\mathrm{e}_\ell}$, we have*

$$\|P_t f\|_{\beta,\mathrm{e}_\ell}\lesssim t^{-\frac{\beta-\alpha}{2}}\|f\|_{\alpha,\mathrm{e}_\ell}\;,$$

uniformly over all ℓ in a compact set of $\mathbf{R}$ and all t in a compact set of $[0,\infty)$.

Proof. We use a decomposition of the heat kernel $P_t(x)=P_+(t,x)+P_-(t,x)$ where P_- is smooth and P_+ is supported in the unit ball centred at 0. Using the decay properties of the heat kernel, the statement regarding P_- is easy to check. Concerning the singular part, one writes $P_+ = \sum_{n\geq 0} P_n$ where each P_n is a smooth function supported in the parabolic annulus $\{(t,x) : 2^{-n-1} \leq |t|^{\frac{1}{2}} + |x| \leq 2^{-n+1}\}$ and such that $P_n(t,x) = 2^{dn}P_0(2^{2n}t, 2^n x)$. Then, we get

$$|\langle f, \eta_x^\lambda(\cdot - y)\rangle| \lesssim \lambda^\alpha \mathrm{e}_\ell(x+y) , \qquad |\langle f, D_x^k P_n(t, \cdot - y)\rangle| \lesssim 2^{-n(\alpha-|k|)}\mathrm{e}_\ell(y) ,$$

uniformly over all $\eta \in \mathcal{B}_1^r$, all $x, y \in \mathbf{R}^d$, all $t > 0$, all $n \geq 0$ and all $k \in \mathbf{N}^2$. Notice that $P_n(t,\cdot)$ vanishes as soon as $n \geq 1 - \frac{1}{2}\log_2 t$. Consequently,

$$|\langle P_+(t) * f, \eta_x^\lambda\rangle| \lesssim \mathrm{e}_\ell(x)(\lambda^\alpha \wedge t^{\frac{\alpha}{2}}) , \quad |\langle f, D_x^k P_+(t, \cdot - x)\rangle| \lesssim \mathrm{e}_\ell(x) t^{\frac{\alpha-|k|}{2}} ,$$

so that the statement follows by interpolation. $\square$

3 Bounds on Y and Z

Let us collect a few facts on the behaviour of smooth functions with a singularity at the origin; we refer to [Hai14, Sec. 10.3] for proofs. For any smooth function $K : \mathbf{R}^d \backslash \{0\} \to \mathbf{R}$ and any real number ζ, we define

$$\|K\|_{\zeta;m} = \sup_{|k|\leq m} \sup_{x\in\mathbf{R}^d} \|x\|^{|k|-\zeta} |D_x^k K(x)| ,$$

where the first supremum runs over $k \in \mathbf{N}^d$ and $|k| = \sum_i k_i$. We say that K is of order ζ if $\|K\|_{\zeta;m} < \infty$ for all $m \in \mathbf{N}$. Recall ϱ_ϵ from the introduction, and define $K_\epsilon = K * \varrho_\epsilon$. If K is of order $\zeta \in (-d, 0)$ then for all $m \in \mathbf{N}$, there exists $C > 0$ such that $\|K_\epsilon\|_{\zeta;m} \leq C\|K\|_{\zeta;m}$, uniformly over $\epsilon \in (0,1]$. Furthermore, for all $\bar\zeta \in [\zeta - 1, \zeta)$, there exists a constant $C > 0$ such that

$$\|K - K_\epsilon\|_{\bar\zeta;m} \leq C\epsilon^{\zeta-\bar\zeta}\|K\|_{\zeta;m} .$$

If K_1 and K_2 are of order ζ_1 and ζ_2 respectively, then K_1K_2 is of order $\zeta = \zeta_1 + \zeta_2$ and we have the bound

$$\|K_1K_2\|_{\zeta;m} \leq C\|K_1\|_{\zeta_1;m}\|K_2\|_{\zeta_2;m} ,$$

where C is a positive constant.

Assume that $\zeta_1 \wedge \zeta_2 > -d$. We set $\zeta = \zeta_1 + \zeta_2 + d$. If $\zeta < 0$, then $K_1 * K_2$ is of order ζ and we have the bound

$$\|K_1 * K_2\|_{\zeta;m} \leq C\|K_1\|_{\zeta_1;m}\|K_2\|_{\zeta_2;m} . \tag{3.1}$$

On the other hand, if $\zeta \in \mathbf{R}_+\backslash\mathbf{N}$ and K_1, K_2 are compactly supported, then the function

$$K(x) = (K_1 * K_2)(x) - \sum_{|k|<\zeta} \frac{x^k}{k!} D_x^k (K_1 * K_2)(0) ,$$

is of order ζ and a bound similar to (3.1) holds, but with the constant C depending on the size of the supports in general.

We will apply these bounds to the function G defined in the introduction. Since G is smooth on $\mathbf{R}^2\backslash\{0\}$, compactly supported and satisfies $G(x) = -\frac{\log|x|}{2\pi}$ in a neighbourhood of the origin, it is a function with a singularity of order ζ, for all $\zeta < 0$, according to our definition. From now on, we set $\varrho^{*2} = \varrho * \varrho$ and we assume without loss of generality that ϱ, ϱ^{*2} are supported in the unit ball of $\mathbf{R}^2$.

Lemma 3.1 *Fix $\kappa \in (0,1)$. We have the bounds*

$$\mathbb{E}\Big[|Z(\eta_x^\lambda)|^2\Big]^{\frac12} \lesssim \lambda^{-\kappa}\;,\;\; \mathbb{E}\Big[|Z_\epsilon(\eta_x^\lambda)|^2\Big]^{\frac12} \lesssim \lambda^{-\kappa}\;,\;\; \mathbb{E}\Big[|Z_\epsilon(\eta_x^\lambda) - Z(\eta_x^\lambda)|^2\Big]^{\frac12} \lesssim \lambda^{-5\kappa}\epsilon^{\kappa}\;,$$

uniformly over all $\epsilon, \lambda \in (0,1)$, all $x \in \mathbf{R}^2$ and all $\eta \in \mathcal{B}_1^r$.

Proof. By translation invariance, it suffices to consider $x = 0$. The random variables $Z(\eta^\lambda)$, $Z_\epsilon(\eta^\lambda)$ and $Z_\epsilon(\eta^\lambda) - Z(\eta^\lambda)$ all belong to the second *homogeneous* Wiener chaos associated with the noise ξ. This is because the constant C_ϵ has been chosen to cancel the 0-th Wiener chaos component of $|\nabla Y_\epsilon|^2$. We start with the second bound of the statement:

$$\begin{aligned}\mathbb{E}\Big[|Z_\epsilon(\eta^\lambda)|^2\Big] &= \sum_{i=1}^2 \int_{z,\tilde z}\Big(\int \eta^\lambda(x) D_{x_i}G_\epsilon(z-x)D_{x_i}G_\epsilon(\tilde z - x)dx\Big)^2 dz\,d\tilde z\\ &= \sum_{i=1}^2 \iint \eta^\lambda(x)\eta^\lambda(x')\Big((D_{x_i}G_\epsilon) * (D_{x_i}G_\epsilon)(x-x')\Big)^2 dx\,dx'\;,\end{aligned}$$

so that the bounds at the beginning of the section yield the desired result. The first bound of the statement follows by replacing G_ϵ by G in the expression above. We turn to the proof of the third bound. To that end, we write

$$\mathbb{E}\Big[|Z_\epsilon(\eta^\lambda) - Z(\eta^\lambda)|^2\Big] = \sum_{i=1}^2 \iint \eta^\lambda(x)\eta^\lambda(x')H_{\epsilon,i}(x-x')dx\,dx'\;,$$

where

$$\begin{aligned}H_{\epsilon,i}(y) = &\Big((D_{x_i}(G_\epsilon - G)) * D_{x_i}G_\epsilon\Big)\cdot\Big((D_{x_i}(G_\epsilon + G)) * D_{x_i}G_\epsilon\Big)(y)\\ &- \Big((D_{x_i}(G_\epsilon - G)) * D_{x_i}G\Big)\cdot\Big((D_{x_i}(G_\epsilon + G)) * D_{x_i}G\Big)(y)\;,\end{aligned}$$

so that, once again, the bounds on the behaviour of singular functions at the origin yield the asserted bound. $\square$

Proof of Proposition 1.3. Let L denote an arbitrary element among Z, Z_ϵ and $Z - Z_\epsilon$. Using the equivalence of moments of elements in inhomogeneous Wiener chaoses of finite order, we obtain

$$\mathbb{E}\left[\sup_{n\ge0}\sup_{x\in\Lambda_n}\Big(\frac{L(\psi_x^n)}{\mathrm{p}_a(x)2^{-n\alpha-n}}\Big)^{2p}\right] \lesssim \sum_{k\in\mathbf{Z}^2}\frac{1}{\mathrm{p}_a(k)^{2p}}\sum_{n\ge0}\sum_{x\in\Lambda_n\cap B(k,1)}\frac{\mathbb{E}[L(\psi_x^n)^2]^p}{2^{-n\alpha2p-2np}}\;.$$

When L is equal to Z or Z_ϵ, Lemma 3.1 ensures that $\mathbb{E}[L(\psi_x^n)^2] \lesssim 2^{-2n+\kappa n}$ uniformly over all x, n, and ϵ. Moreover, $\#(\Lambda_n \cap B(k,1)) \lesssim 2^{2n}$, so that

$$\mathbb{E}\left[\sup_{n\geq 0}\sup_{x\in\Lambda_n}\left(\frac{L(\psi_x^n)}{\mathrm{p}_a(x)2^{-n\alpha-n}}\right)^{2p}\right] \lesssim \sum_{k\in\mathbf{Z}^2}\frac{1}{\mathrm{p}_a(k)^{2p}}\sum_{n\geq 0}2^{np(2\alpha+\kappa)+2n} .$$

This quantity is finite for $\alpha = -\kappa$ and p large enough. Therefore, Z and Z_ϵ belong to $\mathcal{C}^{-\kappa}_{\mathrm{p}_a}$.

Regarding $Z - Z_\epsilon$, Lemma 3.1 ensures that $\mathbb{E}[(Z-Z_\epsilon)(\psi_x^n)^2] \lesssim \epsilon^\kappa 2^{n(5\kappa-2)}$ uniformly over all x, n and ϵ. Then, the same arguments as before yield

$$\mathbb{E}\left[\sup_{n\geq 0}\sup_{x\in\Lambda_n}\left(\frac{(Z-Z_\epsilon)(\psi_x^n)}{\mathrm{p}_a(x)2^{-n\alpha-n}}\right)^{2p}\right] \lesssim \sum_{k\in\mathbf{Z}^2}\frac{1}{\mathrm{p}_a(k)^{2p}}\sum_{n\geq 0}\epsilon^{\kappa p}2^{n(2\alpha p+5\kappa p+2)} ,$$

so that, choosing for instance $\alpha = -3\kappa$ and p large enough, one gets the bound $\mathbb{E}\big[\|Z-Z_\epsilon\|_{-3\kappa,\mathrm{p}_a}\big] \lesssim \epsilon^{\frac{\kappa}{2}}$ uniformly over all $\epsilon \in (0,1]$, thus concluding the proof. □

Proof of Corollary 1.2. Since G is compactly supported and coincides with the Green function of the Laplacian in a neighbourhood of the origin, the classical Schauder estimates [Sim97] imply that for any $\alpha \in \mathbf{R}$, the bounds

$$\|G * f\|_{\alpha+2} \lesssim \|f\|_\alpha , \quad \|D_{x_i} G * f\|_{\alpha+1} \lesssim \|f\|_\alpha ,$$

hold uniformly over all $f \in \mathcal{C}^\alpha$. Recall the functions $\chi_k, k \in \mathbf{Z}^d$ from the proof of Theorem 2.6. Since G is compactly supported, we deduce from the bounds above that

$$\|G * (f\chi_k)\|_{\alpha+2} \lesssim w(k)\|f\|_{\alpha,w} , \quad \|D_{x_i} G * (f\chi_k)\|_{\alpha+1} \lesssim w(k)\|f\|_{\alpha,w} ,$$

uniformly over all $k \in \mathbf{Z}^d$ and all $f \in \mathcal{C}^\alpha_w$. For fixed x, only a bounded number of $\{\chi_k(x), k \in \mathbf{Z}^d\}$ are non-zero, uniformly over all $x \in \mathbf{R}^d$. Since $f = \sum_{k\in\mathbf{Z}^d} f\chi_k$, we deduce that

$$\|G * f\|_{\alpha+2,w} \lesssim \|f\|_{\alpha,w} , \quad \|D_{x_i} G * f\|_{\alpha+1,w} \lesssim \|f\|_{\alpha,w} ,$$

uniformly over all $f \in \mathcal{C}^\alpha_w$. This being given, the statement is a direct consequence of Lemma 1.1. □

4 Picard iteration

Fix $\kappa \in (0,\frac{1}{2})$, and let the parameter a appearing in the weight p_a be any value in $(0,\frac{\kappa}{2})$. Let $g, h^{(1)}, h^{(2)} \in \mathcal{C}^{-\kappa}_{\mathrm{p}_a}$ and $f \in \mathcal{C}^{-1+4\kappa}_{e_\ell}$ be given. We define the map $v \mapsto \mathcal{M}_{T,f}(v)$ as follows:

$$\mathcal{M}_{T,f}(v)_t = \int_0^t P_{t-s} * \Big(v_s \cdot g + D_{x_i} v_s \cdot h^{(i)}\Big) ds + P_t * f .$$

In this equation, there is an implicit summation over $i \in \{1,2\}$. This convention will be in force for the rest of the article.

Proposition 4.1 *For any given* $g, h^{(1)}, h^{(2)} \in \mathcal{C}^{-\kappa}_{\mathrm{p}_a}$ *and any* $f \in \mathcal{C}^{-1+4\kappa}_{\mathrm{e}_{\ell_0}}$*, the map* $\mathcal{M}_{T,f}$ *admits a unique fixed point* $v \in \mathcal{E}^{1+2\kappa}_{\ell_0,T}$*. Furthermore, the solution map* $(g, h^{(1)}, h^{(2)}, f) \mapsto v$ *is continuous.*

Proof. First, Lemma 2.7 ensures that $\|P_t * f\|_{1+2\kappa,\mathrm{e}_{\ell+t}} \lesssim t^{-1+\kappa}\|f\|_{-1+4\kappa,\mathrm{e}_\ell}$ uniformly over all t in any given compact interval of $\mathbf{R}_+$. Second, using Theorem 2.6 and the simple inequality

$$\sup_{x\in\mathbf{R}^2} \frac{\mathrm{p}_a(x)\mathrm{e}_{\ell+s}(x)}{\mathrm{e}_{\ell+t}(x)} \le e^{-a}\Big(\frac{a}{t-s}\Big)^a ,$$

we obtain

$$\begin{aligned}\|v_s \cdot g + D_{x_i} v_s \cdot h^{(i)}\|_{-\kappa,\mathrm{e}_{\ell+t}} &\lesssim (t-s)^{-a}\|v_s\|_{1+2\kappa,\mathrm{e}_{\ell+s}}(||g||_{-\kappa,\mathrm{p}_a} + ||h^{(i)}||_{-\kappa,\mathrm{p}_a})\\ &\lesssim (t-s)^{-a}s^{-1+\kappa}\|\!|v|\!\|_{\ell,T}(||g||_{-\kappa,\mathrm{p}_a} + ||h^{(i)}||_{-\kappa,\mathrm{p}_a}) ,\end{aligned}$$

uniformly over all s, t in a compact set of $\mathbf{R}_+$ and all ℓ in a compact set of $\mathbf{R}$. Then, by Lemma 2.7 and using $a < \kappa/2$, we obtain

$$\begin{aligned}&\Big\|\int_0^t P_{t-s} * \Big(v_s \cdot g + D_{x_i} v_s \cdot h^{(i)}\Big)\, ds\Big\|_{1+2\kappa,\mathrm{e}_{\ell+t}} \qquad (4.1)\\ &\quad\lesssim \int_0^t (t-s)^{-\frac12-2\kappa}s^{-1+\kappa}ds\, \|\!|v|\!\|_{\ell,T}(||g||_{-\kappa,\mathrm{p}_a} + ||h^{(i)}||_{-\kappa,\mathrm{p}_a})\\ &\quad\lesssim t^{-1+\kappa}T^{\frac12-2\kappa}\|\!|v|\!\|_{\ell,T}(||g||_{-\kappa,\mathrm{p}_a} + ||h^{(i)}||_{-\kappa,\mathrm{p}_a}) ,\end{aligned}$$

uniformly over all $t \in (0, T]$. This ensures that $\mathcal{M}_{T,f}(v) \in \mathcal{E}^{1+2\kappa}_{\ell,T}$. Furthermore we have

$$\|\!|\mathcal{M}_{T,f}(v) - \mathcal{M}_{T,f}(\bar v)|\!\|_{\ell,T} \lesssim T^{\frac12-2\kappa}\|\!|v-\bar v|\!\|_{\ell,T}(\|g\|_{-\kappa,\mathrm{p}_a} + \|h^{(i)}\|_{-\kappa,\mathrm{p}_a}) , \quad (4.2)$$

uniformly over all ℓ in a compact set of $\mathbf{R}$, all T in a compact set of $\mathbf{R}_+$, all $f \in \mathcal{C}^{-1+4\kappa}_{e_\ell}$ and all $v, \bar v \in \mathcal{E}_{\ell,T}$. (Here and below we write $\mathcal{E}_{\ell,T}$ instead of $\mathcal{E}^{1+2\kappa}_{\ell,T}$ for conciseness.) Consequently, there exists $T^* > 0$ such that $\mathcal{M}_{T^*,f}$ is a contraction on $\mathcal{E}_{\ell,T^*}$, uniformly over all $\ell \in [\ell_0, \ell_0 + T]$ and all $f \in \mathcal{C}^{-1+4\kappa}_{e_\ell}$. Fix an initial condition $f \in \mathcal{C}^{-1+4\kappa}_{e_{\ell_0}}$. To obtain a fixed point for the map $\mathcal{M}_{T,f}$, we proceed by iteration. The map $\mathcal{M}_{T^*,f}$ admits a unique fixed point $v^* \in \mathcal{E}_{\ell_0,T^*}$. If $T^* \ge T$, we are done. Otherwise, set $f^* := v^*_{T^*/2} \in \mathcal{C}^{1+2\kappa}_{e_{\ell_0^*}}$, where $\ell_0^* = \ell_0 + T^*/2$. Since $\ell_0^* \le \ell_0 + T$, the map $\mathcal{M}_{T^*,f^*}$ is again a contraction on $\mathcal{E}_{\ell_0^*,T^*}$, so that it admits a unique fixed point $v^{**} \in \mathcal{E}_{\ell_0^*,T^*}$. We define $v_s := v^*_s$ for all $s \in (0, T^*/2]$ and $v_s := v^{**}_{s-T^*/2}$ for all $s \in (T^*/2, 3T^*/2]$. A simple calculation shows that v is a fixed point of $\mathcal{M}_{\frac{3T^*}{2},f}$ and that $v \in \mathcal{E}_{\ell_0,3T^*/2}$. Suppose that $\bar v$ is another fixed point. By the uniqueness of the fixed point on $(0, T^*]$, we deduce that v^* and $\bar v$ coincide on this interval. Moreover, a simple calculation shows that $(\bar v_{s+\frac{T^*}{2}}, s \in (0, T^*])$ is necessarily a fixed point of $\mathcal{M}_{T^*,f^*}$ so that it coincides with v^{**}. Iterating this argument ensures existence and uniqueness of the fixed point on any interval $[0, T]$.

We turn to the continuity of the solution map with respect to f, g and $h^{(i)}$. Let $\bar{\mathcal{M}}$ be the map associated with $\bar{g}$ and $\bar{h}^{(i)}$. For any initial conditions f and $\bar{f}$ in $\mathcal{C}_{\mathrm{e}_\ell}^{-1+4\kappa}$, both $\mathcal{M}_{T,f}$ and $\bar{\mathcal{M}}_{T,\bar{f}}$ admit a unique fixed point v and $\bar{v}$. Furthermore, we have

$$\begin{aligned} v_t - \bar{v}_t = &\Big(\mathcal{M}_{T,f}(v) - \mathcal{M}_{T,f}(\bar{v})\Big)_t + \int_0^t P_{t-s} * \Big(\bar{v}_s(g-\bar{g}) + D_{x_i}\bar{v}_s(h^{(i)} - \bar{h}^{(i)})\Big) ds \\ &+ P_t * (f - \bar{f}) \;. \end{aligned}$$

Using (4.1) and (4.2), we deduce that

$$\begin{aligned} \|v - \bar{v}\|_{\ell,T} \lesssim\; & T^{\frac{1}{2}-2\kappa} \|v - \bar{v}\|_{\ell,T} (\|g\|_{-\kappa,\mathrm{p}_a} + \|\bar{g}\|_{-\kappa,\mathrm{p}_a} + \|h^{(i)}\|_{-\kappa,\mathrm{p}_a} + \|\bar{h}^{(i)}\|_{-\kappa,\mathrm{p}_a}) \\ &+ T^{\frac{1}{2}-2\kappa} \|v\|_{\ell,T} (\|\bar{g} - g\|_{-\kappa,\mathrm{p}_a} + \|\bar{h}^{(i)} - h^{(i)}\|_{-\kappa,\mathrm{p}_a}) \\ &+ \|f - \bar{f}\|_{-1+4\kappa,\ell} \;, \end{aligned}$$

uniformly over all ℓ in a compact set of $\mathbf{R}$ and all T in a compact set of $\mathbf{R}_+$. Fix $R > 0$. There exists $T > 0$ such that

$$\|v - \bar{v}\|_{\ell,T} \lesssim \|f - \bar{f}\|_{-1+4\kappa,\ell} + T^{\frac{1}{2}-2\kappa} (\|\bar{g} - g\|_{-\kappa,\mathrm{p}_a} + \|\bar{h}^{(i)} - h^{(i)}\|_{-\kappa,\mathrm{p}_a}) \;,$$

uniformly over all ℓ in a compact set of $\mathbf{R}$ and all $g, \bar{g}, h, \bar{h}$ such that $\|v\|_{\ell,T}$, $\|g\|_{-\kappa,\mathrm{p}_a}$, $\|\bar{g}\|_{-\kappa,\mathrm{p}_a}$, $\|h^{(i)}\|_{-\kappa,\mathrm{p}_a}$ and $\|\bar{h}^{(i)}\|_{-\kappa,\mathrm{p}_a}$ are smaller than R. This yields the continuity of the solution map on $(0, T]$. By iterating the argument as above, we obtain continuity on any bounded interval. $\square$

We are now in position to prove the main result of this article.

Proof of Theorem 1.4. Let u_0 be an element in $\mathcal{C}_{\mathrm{e}_\ell}^{-1+4\kappa}$ for a given $\ell \in \mathbf{R}$. Let $f_\epsilon := u_0 e^{Y_\epsilon}$. By Corollary 1.2 and Theorem 2.6, f_ϵ converges to $f = u_0 e^Y$ in $\mathcal{C}_{\mathrm{e}_{\ell''}}^{-1+4\kappa}$ for any $\ell'' > \ell$. Let v_ϵ be the unique fixed point of $\mathcal{M}_{T,f_\epsilon}$ with $g_\epsilon = Z_\epsilon - F * \xi_\epsilon$ and $h_\epsilon^{(i)} := -2D_{x_i} Y_\epsilon$. By Corollary 1.2 and Proposition 1.3, we know that g_ϵ, $h_\epsilon^{(i)}$ converge in probability to

$$g = Z - F * \xi \;, \qquad h^{(i)} = -2D_{x_i} Y \;,$$

in $\mathcal{C}_{\mathrm{p}_a}^{-\kappa}$. Notice that the convergence of $F * \xi_\epsilon$ towards $F * \xi$ is a consequence of Lemma 1.1, since F is a compactly supported, smooth function. Therefore, Proposition 4.1 ensures that v_ϵ converges in probability in $\mathcal{E}_{\ell'',T}^{1+2\kappa}$ to the unique fixed point v of the map $\mathcal{M}_{T,f}$ associated to $g, h^{(1)}, h^{(2)}$. Moreover, Theorem 2.6 ensures that, for any $\ell' > \ell''$, $u_\epsilon = v_\epsilon e^{-Y_\epsilon}$ converges to $u = v e^{-Y}$ in the space $\mathcal{E}_{\ell',T}^{1-\kappa}$. $\square$

References

[BCD11] H. BAHOURI, J.-Y. CHEMIN, and R. DANCHIN. *Fourier analysis and nonlinear partial differential equations*, vol. 343 of *Grundlehren der Mathematischen Wissenschaften*. Springer, Heidelberg, 2011.

[CJK13] D. CONUS, M. JOSEPH, and D. KHOSHNEVISAN. On the chaotic character of the stochastic heat equation, before the onset of intermitttency. *Ann. Probab.* **41**, no. 3B, (2013), 2225–2260. doi:10.1214/11-AOP717.

[CM94] R. A. CARMONA and S. A. MOLCHANOV. Parabolic Anderson problem and intermittency. *Mem. Amer. Math. Soc.* **108**, no. 518, (1994), viii+125. doi:10.1090/memo/0518.

[Dau88] I. DAUBECHIES. Orthonormal bases of compactly supported wavelets. *Comm. Pure Appl. Math.* **41**, no. 7, (1988), 909–996. doi:10.1002/cpa.3160410705.

[DPD02] G. DA PRATO and A. DEBUSSCHE. Two-dimensional Navier-Stokes equations driven by a space-time white noise. *J. Funct. Anal.* **196**, no. 1, (2002), 180–210. doi:10.1006/jfan.2002.3919.

[GIP12] M. GUBINELLI, P. IMKELLER, and N. PERKOWSKI. Paracontrolled distributions and singular PDEs. *ArXiv e-prints* (2012). arXiv:1210.2684v3.

[Hai14] M. HAIRER. A theory of regularity structures. *Invent. Math.* **198**, no. 2, (2014), 269–504. arXiv:1303.5113. doi:10.1007/s00222-014-0505-4.

[HP14] M. HAIRER and É. PARDOUX. A Wong-Zakai theorem for stochastic PDEs. *ArXiv e-prints* (2014). arXiv:1409.3138. J. Math. Soc. Japan, to appear.

[HPP13] M. HAIRER, E. PARDOUX, and A. PIATNITSKI. Random homogenisation of a highly oscillatory singular potential. *Stoch. PDEs: Anal. Comp.* **1**, no. 4, (2013), 571–605. arXiv:1303.1955. doi:10.1007/s40072-013-0018-y.

[Mey92] Y. MEYER. *Wavelets and operators*, vol. 37 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1992. Translated from the 1990 French original by D. H. Salinger.

[Nua06] D. NUALART. *The Malliavin calculus and related topics*. Probability and its Applications (New York). Springer-Verlag, Berlin, second ed., 2006.

[Sim97] L. SIMON. Schauder estimates by scaling. *Calc. Var. Partial Differential Equations* **5**, no. 5, (1997), 391–407. doi:10.1007/s005260050072.