The motion of a random string

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Abstract

We review a series of forthcoming results leading to the construction of a natural evolution on the space of loops with values in a Riemannian manifold. In particular, this clarifies the algebraic structure of the renormalisation procedures appearing in the context of the theory of regularity structures.

1 Introduction

This work is partially motivated by Funaki's attempt [Fun88, Fun92] to construct a natural evolution on the space of loops with values in a manifold. Given a compact smooth Riemannian manifold $\mathscr{M}$ with metric g, write $\mathscr{L}^\infty\mathscr{M}$ for the space of all smooth loops in $\mathscr{M}$, i.e. simply the space of all smooth functions $u\colon S^1 \to \mathscr{M}$. We also write $\mathscr{L}\mathscr{M}$ for the space of all loops, where we only impose that u is continuous. The energy of a loop $u \in \mathscr{L}^\infty\mathscr{M}$ is given by

$$E(u) = \frac{1}{2}\int_{S^1} g_{u(x)}(\partial_x u(x), \partial_x u(x))\,dx\ . \tag{1.1}$$

The aim of this note is to discuss the construction of the natural Langevin dynamic associated to E. In other words, one would like to build a Markov process u with values in $\mathscr{L}\mathscr{M}$ with the following two properties:

1. The measure on loops formally given by $\exp(-2E(u))\,Du$ is invariant for the process u.
2. The evolution is local in the sense that, when written as an evolution equation, all of the terms appearing in its right hand side are purely local.

The meaning of the first property is not clear a priori since there is of course no "Lebesgue measure Du on loop space", but a natural way of interpreting it is as the Brownian bridge measure on $\mathscr{M}$. See [IM85, AD99] for proofs that natural

approximations of $\exp(-2E(u))\,Du$ do indeed converge to the Brownian bridge measure, or to close relatives thereof. The second property is natural in view of the fact that the integrand appearing in (1.1) depends on u in a local way.

1.1 Derivation of the main equation

Recall that in the finite-dimensional case, given a smooth potential $V\colon \mathbf{R}^d \to \mathbf{R}$ with sufficiently "nice" behaviour at infinity, the Langevin dynamic associated to V is given by the stochastic differential equation

$$\dot{x} = -\nabla V(x) + \xi\,, \tag{1.2}$$

where ξ denotes white noise, i.e. the generalised $\mathbf{R}^d$-valued stochastic process with

$$\mathbf{E}\xi_i(s)\xi_j(t) = \delta_{ij}\delta(t-s)\,. \tag{1.3}$$

In order to mimic this, we therefore first need a natural notion of gradient for E. For this, note that the metric g induces a Riemannian structure $\hat{g}$ on $\mathscr{L}^\infty\mathscr{M}$ in the following way. An element $h \in T_u\mathscr{L}^\infty\mathscr{M}$ of the tangent space of $\mathscr{L}^\infty\mathscr{M}$ at $u \in \mathscr{L}^\infty\mathscr{M}$ is given by a function $h\colon S^1 \to T\mathscr{M}$ is such that $h(x) \in T_{u(x)}\mathscr{M}$ for every $x \in S^1$. One then defines a "Riemannian metric" $\hat{g}$ on $T\mathscr{L}^\infty\mathscr{M}$ by setting

$$\hat{g}_u(h,h) = \int_{S^1} g_{u(x)}(h(x),h(x))\,dx\,,$$

for every $h \in T_u\mathscr{L}^\infty\mathscr{M}$. As usual, one can then define the gradient $\nabla E(u)$ as the unique element of $T_u\mathscr{L}^\infty\mathscr{M}$ such that

$$\hat{g}_u(h, \nabla E(u)) = dE(h)\,,$$

for every $h \in T_u\mathscr{L}^\infty\mathscr{M}$, where $dE_u(h)$ is the differential of E in the direction h evaluated at u. This yields a particular case of the Eells-Sampson Laplacian [ES64].

In view of (1.2), it is then natural to try to define the Langevin equation associated to (1.1) as

$$\dot{u} = -\nabla E(u) + \xi\,, \tag{1.4}$$

where ξ is a suitable "white noise on loop space". In particular, in the simplest possible case where $\mathscr{M} = \mathbf{R}$ (with its standard metric), one obtains the stochastic heat equation $\dot{u} = \partial_x^2 u + \xi$, for ξ a space-time white noise, the invariant measure of which is indeed given by the Brownian bridge. In general however, it is much less clear what the symbol ξ appearing in (1.4) latter actually means since at any fixed time ξ should belong to $T_u\mathscr{L}\mathscr{M}$ (or at least a suitable generalisation thereof), which itself varies with time.

Note that a characterisation of the noise ξ appearing in (1.2) is that it is the (generalised) Gaussian process with Cameron-Martin space given by $L^2(\mathbf{R}_+, \mathbf{R}^d)$ endowed with its usual scalar product. This hints at a way of adapting this definition to the situation at hand. Indeed, for any given smooth function $u\colon \mathbf{R}_+ \to \mathscr{L}^\infty\mathscr{M}$ (which of course the process u we are interested in really isn't!), we obtain a scalar product on the space L^2_u of functions $h\colon \mathbf{R}_+ \to T\mathscr{L}^\infty\mathscr{M}$ with $h(t) \in T_{u(t)}\mathscr{L}\mathscr{M}$ by

$$\langle h, \bar{h}\rangle_u = \int_{\mathbf{R}} \int_{S^1} \langle h(t,x), \bar{h}(t,x)\rangle_{u(t,x)}\, dx\, dt\ . \tag{1.5}$$

then a natural definition for ξ mimicking (1.3) is that it is the (generalised) Gaussian process with Cameron-Martin space given by L^2_u endowed with the scalar product (1.5). It turns out that this process can be constructed more explicitly as follows. Take a finite collection $\{\sigma_i\}_{i=1}^m$ of vector fields on $\mathscr{M}$ with the property that, for every $u \in \mathscr{M}$ and $h, \bar{h} \in T_u\mathscr{M}$, one has the identity

$$g_u(h, \bar{h}) = \sum_{i=1}^m g_u(h, \sigma_i(u)) g_u(\bar{h}, \sigma_i(u))\ ,$$

and let $\{\xi_i\}_{i=1}^m$ be a collection of independent space-time white noises. Then, one can verify that the Gaussian process ξ given by $\xi(t,x) = \sum_{i=1}^m \sigma_i(u(t,x))\, \xi_i(t,x)$ does indeed have as its Cameron-Martin space L^2_u with scalar product (1.5).

This suggests that (1.4) should really be interpreted as

$$\dot{u} = -\nabla E(u) + \sum_{i=1}^m \sigma_i(u)\, \xi_i\ .$$

In local coordinates, this can be written as

$$\dot{u}^\alpha = \partial_x^2 u^\alpha + \Gamma^\alpha_{\beta\gamma}(u)\, \partial_x u^\beta \partial_x u^\gamma + \sigma_i^\alpha(u)\, \xi_i\ , \tag{1.6}$$

where Einstein's convention of summation over repeated indices is implied and $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols for the Levi-Civita connection of $(\mathscr{M}, g)$. We thus derived some kind of multi-component version of the KPZ equation, similar to the one studied in [FSS13, Spo14], but with both Γ and σ depending on the solution u itself. This also shows where the problem lies. Since we expect the stationary measure of (1.6) to be closely related to the Brownian bridge, we expect typical solutions to be not even $\frac{1}{2}$-Hölder continuous as a function of the x coordinate. As a consequence, the term $\partial_x u^\beta \partial_x u^\gamma$ involves the product of two distributions, so that it is not clear *a priori* how to define it. Even if we were to somehow define it, it still wouldn't be clear how to multiply the resulting distribution with the very irregular function $\Gamma^\alpha_{\beta\gamma}(u)$.

Similarly, while one would expect to be able to define the terms $\sigma_i^\alpha(u)\,\xi_i$ via Itô integration, this would certainly lead to problems since Itô integration famously does not preserve the chain rule, which makes it ill-suited for problems on manifolds. On the other hand, there appears to be no Stratonovich integration in this context because the corresponding Itô correction term is infinite, see for example [HP15].

1.2 Main result

In the present work, we take a somewhat pragmatic approach by postulating that we call a process a "solution" to (1.6) if it is the limit of a natural approximation procedure. More precisely, we fix a function $\varrho \in \mathscr{C}_0^\infty(\mathbf{R}^2)$ integrating to 1 and we set

$$\xi_i^{(\varepsilon)} = \varrho^{(\varepsilon)} \star \xi\ , \qquad \varrho^{(\varepsilon)}(t,x) = \varepsilon^{-3}\varrho(t/\varepsilon^2, x/\varepsilon)\ . \tag{1.7}$$

It is then natural to ask whether solutions to (1.6) with ξ replaced by $\xi^{(\varepsilon)}$ admit limits as $\varepsilon \to 0$. Of course, even if this turns out to be the case, one might lose uniqueness, but as we will see shortly this can be recovered to some extent. In order to formulate our result, we impose an additional constraint on the vector fields σ_i, namely we assume that

$$\sum_{i=1}^m \nabla_{\sigma_i}\sigma_i = 0\ ,$$

where ∇ denotes the covariant derivative (with respect to the Levi-Civita connection). An equivalent way of formulating this is that, viewing the σ_i as first-order differential operators, $\sum_i \sigma_i^2$ is equal to the Laplace-Beltrami operator. This should be viewed as a "centering" condition, which can always be satisfied. For example, it suffices to consider an isometric embedding of $\mathscr{M}$ as a submanifold of $\mathbf{R}^m$ and to then take for $\sigma_i(u)$ the orthogonal projection of the ith canonical basis vector of $\mathbf{R}^m$ onto $T_u\mathscr{M}$.

The main result on which we report here is a consequence of the more general results exposed in [BHZ16, CH16], relying on [Hai14, HQ15], and can be formulated as follows.

Theorem 1.1 *Let $u^{(\varepsilon)}$ be the solution to* (1.6) *with ξ_i replaced by $\xi_i^{(\varepsilon)}$ and with some fixed Hölder continuous initial condition u_0. Then, as $\varepsilon \to 0$, $u^{(\varepsilon)}$ converges in probability to a Markov process u. Furthermore, although u might depend on the choice of ϱ, the set of possible limits can be parametrised smoothly by finitely many parameters.*

Remark 1.2 If $\mathscr{M}$ is a symmetric space, then the limit can be shown to be independent of ϱ. In general, we believe that the set of possible limits for approximations of the type discussed here is actually one-dimensional, but we do

not have a proof of this. We also believe that the law of the limit is independent of the choice of vector fields σ_i, but again we have no proof of this.

Remark 1.3 A natural process on loop space leaving invariant the Brownian bridge measure on a manifold was constructed by a number of authors in the nineties, see for example [Dri92, DR92, Dri94, ES96, Nor98]. This process is different since, in the particular case of $\mathscr{M} = \mathbf{R}^d$, it would correspond to the Ornstein-Uhlenbeck process from Malliavin calculus, rather than the stochastic heat equation. In particular, it is not "local" in the sense that its driving noise has non-trivial spatial correlations.

To some extent, we can probably learn more from the proof of Theorem 1.1 than from the result itself, although a more systematic study of the properties of the limiting process, especially in the small noise limit, could be very interesting, especially in view of [Wit82, Biso8]. We will assume in the sequel that the reader has at least some familiarity with the theory of regularity structures as developed in [Hai14] and surveyed for example in [Hai15, CW15]. The proof follows the same methodology as developed in [Hai14, HQ15, HP15], which can be summarised as follows:

1. Build a regularity structure $\mathscr{T} = (\mathcal{T}, \mathcal{G}_+)$ that is sufficiently rich to be able to formulate (1.6) as a fixed point problem in some space $\mathscr{D}^{\gamma,\eta}$ of sufficiently regular modelled distributions and build a local solution theory.
2. Exhibit a sufficiently large subgroup $\mathcal{G}_-$ of the "renormalisation group" $\mathfrak{R}$ associated to $\mathscr{T}$ as in [Hai14, Def. 8.41].
3. Find a sequence of elements $M_\varepsilon \in \mathcal{G}_-$ such that the sequence of admissible models $\hat{\Pi}^{(\varepsilon)}$ obtained by taking the canonical lift $\Pi^{(\varepsilon)}$ of $\{\xi_i^{(\varepsilon)}\}_{i=1}^m$ and acting on it with M_ε converges to a limiting model $\hat{\Pi}$.
4. Show that M_ε can be chosen in such a way that the solution to the fixed point problem constructed in 1. with model $\hat{\Pi}^{(\varepsilon)}$ is the same (once higher-order information is discarded) as that of (1.6) with ξ_i replaced by $\xi_i^{(\varepsilon)}$.

It follows immediately from the main results of [Hai14] that if these four steps can be carried out successfully, then Theorem 1.1 follows.

Let us now review how these steps can be performed in the context of the problem at hand. Step 1. in this list is generic and was performed in [Hai14] for a large class of stochastic PDEs which in particular include (1.6), so we will not dwell on it. Step 2. is purely algebraic. The problem is that while elements of $\mathfrak{R}$ are described by linear maps on $\mathcal{T}$ and one can easily make an "educated guess" of a group $\mathcal{G}_-$ of such linear maps that is relevant later on, checking whether a given map belongs to $\mathfrak{R}$ using the characterisation given in [Hai14] is tedious. In the case of [Hai14, HQ15, HP15], the relevant groups were of very low dimension, so that this

verification could easily be done by brute force. In the present situation, even after taking simplifications arising from symmetries into account, the dimension of $\mathcal{G}_-$ is still 71, so that such a brute force verification would only be practical with the help of a computer. In [BHZ16], we therefore obtain a general result showing that those linear maps arising from the "educated guesses" alluded to earlier do indeed always belong to $\mathfrak{R}$, and a presentation of this result is the content of Section 2. Step 3. on the other hand is purely analytical. Again, in the abovementioned previous works, this step was reduced to moment bounds for a finite number (typically about 5 or 6) of stochastic processes, and these were then obtained separately for each of these processes. Again, this could in principle be done here but would be impractical since the number of processes that would require moment bounds in this case is well over one hundred. In the article [CH16], we therefore obtain a general estimate that yields suitable moment bounds for a very large class of processes of this type, thus bypassing the need to consider these processes separately. This bound is presented in Section 3. At this stage, without any additional information but exploiting the $x \leftrightarrow -x$ symmetry of the equation, one deduces from the general theory of [Hai14] that there exist *finitely many* ε-dependent constants $c_j^{(\varepsilon)}$ and ε-independent vector fields h_j such that the solutions to

$$\dot{u}^\alpha = \partial_x^2 u^\alpha + \Gamma^\alpha_{\beta\gamma}(u)\,\partial_x u^\beta \partial_x u^\gamma + c_j^{(\varepsilon)} h_j^\alpha(u) + \sigma_i^\alpha(u)\,\xi_i^{(\varepsilon)}\,,$$

converge to a limit u that is furthermore independent of the mollifier ϱ. In order to complete the proof of Theorem 1.1, it therefore remains to show that for every j, either $c_j^{(\varepsilon)}$ converges to a finite limit as $\varepsilon \to 0$ or $h_j = 0$. This result is the content of forthcoming work and will not be reviewed presently.

2 Algebraic aspects

Recall that the regularity structure canonically associated to (1.6) comprises symbols $\{\Xi_i\}_{i=1}^m$, $\{X^k\}_{k\in\mathbf{N}^2}$ representing the driving noises and Taylor monomials respectively, as well as abstract integration maps $\{\mathcal{I}_k\}_{k\in\mathbf{N}^2}$ representing the heat kernel and its derivatives, and an associative and commutative product. We write $\mathcal{F}$ for the free vector space generated by all formal expressions built from these operations, modulo the usual identifications, namely that $\mathbf{1} \stackrel{\text{def}}{=} X^0$ is neutral for the product, that $X^k X^\ell = X^{k+\ell}$, and that $\mathcal{I}_\ell(X^k) = 0$. Writing $|k| = 2k_0 + k_1$ for $k \in \mathbf{N}^2$, we assign real-valued degrees to these objects by

$$|\Xi_i| = -\frac{3}{2} - \kappa\,, \qquad |X^k| = |k|\,, \qquad |\mathcal{I}_\ell| = 2 - |\ell|\,, \tag{2.1}$$

respectively, for $\kappa > 0$ a parameter with a sufficiently small value ($\kappa = 1/100$ will do). We will use the shorthand notations $\mathcal{I}' = \mathcal{I}_{(0,1)}$, $X_0 = X^{(1,0)}$ and $X_1 = X^{(0,1)}$.

The fact that the time coordinate X_0 has degree 2 rather than 1 reflects the fact that we endow space-time with the parabolic scaling, which is consistent with the scaling used in (1.7).

Of course, not all the formal expressions built from these symbols and operations are useful for our problem. As in [Hai14], we define collections $\mathcal{U}$, $\mathcal{U}'$ and $\mathcal{W}$ of formal expressions as the smallest collections such that $\{\Xi_i\}_{i=1}^m \cup \{X^k\}_{k\in\mathbf{N}^2} \subset \mathcal{W}$, such that

$$\tau \in \mathcal{W} \qquad \Rightarrow \qquad \mathcal{I}(\tau) \in \mathcal{U} \quad \& \quad \mathcal{I}'(\tau) \in \mathcal{U}' \, ,$$

and such that, for every $k \geq 0$, every collection $\{\tau_i\}_{i=1}^k$ with $\tau_i \in \mathcal{U}$ for every i, and every $\sigma_1, \sigma_2 \in \mathcal{U}'$, one has

$$\{\tau, \tau\Xi_j, \tau\sigma_1, \tau\sigma_1\sigma_2\} \subset \mathcal{W} \, , \qquad \tau = \tau_1 \cdots \tau_k \, .$$

Note that by construction one has $\mathcal{U} \cup \mathcal{U}' \subset \mathcal{W}$.

The space $\mathcal{T}$ is then defined as the free vector space generated by $\mathcal{W}$, endowed with the $\mathbf{R}$-grading determined by (2.1). As long as $\kappa < \frac{1}{2}$, one can verify that, for every $\gamma \in \mathbf{R}$, the subspace of $\mathcal{T}$ generated by elements of degree at most γ is finite-dimensional. In order to describe $\mathcal{G}_+$, we consider the free vector space $\hat{\mathcal{T}}_+$ generated by all formal expressions of the type $X^\ell \prod_{i=1}^N \mathcal{I}_{k_i}(\tau_i)$ for some $N \geq 0$, some $\ell, k_i \in \mathbf{N}^2$ and some $\tau_i \in \mathcal{W}$. Note that $\hat{\mathcal{T}}_+$ is an algebra, which is not the case for $\mathcal{T}$ since the product of two expressions in $\mathcal{W}$ does not necessarily belong to $\mathcal{W}$. We then write $\mathcal{T}_+$ for the quotient of $\hat{\mathcal{T}}_+$ by the ideal $\mathcal{I}_- \subset \hat{\mathcal{T}}_+$ generated by all $\mathcal{I}_k(\tau)$ with $|\mathcal{I}_k(\tau)| \leq 0$.

Consider now the linear map $\Delta^+ : \mathcal{T} \to \mathcal{T} \otimes \mathcal{T}_+$ given by

$$\Delta^+ X_i = X_i \otimes \mathbf{1} + \mathbf{1} \otimes X_i \, , \qquad \Delta^+ \Xi_i = \Xi_i \otimes \mathbf{1} \, , \tag{2.2}$$

and then recursively by

$$\Delta^+ \mathcal{I}_k(\tau) = (\mathcal{I}_k \otimes \mathrm{id})\Delta^+ \tau + \sum_{\ell \in \mathbf{N}^2} \frac{X^\ell}{\ell!} \otimes \mathcal{I}_{k+\ell}(\tau) \, , \qquad \Delta^+(\tau\bar{\tau}) = \Delta^+\tau \, \Delta^+\bar{\tau} \, . \tag{2.3}$$

The sum appearing in this expression is actually finite since all but finitely many of the summands are zero in $\mathcal{T} \otimes \mathcal{T}_+$. Note also that we can define $\Delta^+ : \mathcal{T}_+ \to \mathcal{T}_+ \otimes \mathcal{T}_+$ in formally exactly the same way (and this is compatible with the quotienting procedure used to produce $\mathcal{T}_+$), except that in that case the maps $\mathcal{I}_k$ appearing in (2.3) are interpreted as linear maps from $\mathcal{T}$ to $\mathcal{T}_+$. The following was shown in [Hai14, Sec. 8].

Proposition 2.1 *There exists an algebra morphism $\mathcal{A}_+ : \mathcal{T}_+ \to \mathcal{T}_+$ so that $(\mathcal{T}_+, \cdot, \Delta^+, \mathcal{A}_+)$ is a Hopf algebra, and the map Δ^+, viewed as a map $\mathcal{T} \to \mathcal{T} \otimes \mathcal{T}_+$, turns $\mathcal{T}$ into a right comodule for $\mathcal{T}_+$.*

The morphism $\mathcal{A}_+$ is defined uniquely by the fact that $\mathcal{A}_+ X_i = -X_i$ and

$$\mathcal{A}_+ \mathcal{I}_k(\tau) = -\sum_{\ell \in \mathbf{N}^2} \frac{(-X)^\ell}{\ell!} \mathcal{M}_+ (\mathcal{I}_{k+\ell} \otimes \mathcal{A}_+) \Delta^+ \tau \,, \tag{2.4}$$

where $\mathcal{M}_+ \colon \mathcal{T}_+ \otimes \mathcal{T}_+ \to \mathcal{T}_+$ denotes the product.

Denote by $\mathcal{G}_+$ the group of characters $g \colon \mathcal{T}_+ \to \mathbf{R}$ endowed with

$$f \circ g = (f \otimes g)\Delta^+ \,, \qquad f^{-1} = f\mathcal{A}_+ \,, \tag{2.5}$$

as well as the identity element e given by $e(\mathbf{1}) = 1$ and $e(\tau) = 0$ for symbols $\tau \neq \mathbf{1}$. The comodule structure of $\mathcal{T}$ for $\mathcal{T}_+$ then yields a natural action of $\mathcal{G}_+$ onto $\mathcal{T}$ from the left via

$$g \mapsto \Gamma_g \,, \qquad \Gamma_g = (\mathrm{id} \otimes g)\Delta^+ \,.$$

Since furthermore Δ^+ preserves total degree and every basis vector in $\mathcal{T}_+$ has strictly positive degree except for $\mathbf{1}$, this yields a regularity structure $\mathscr{T} = (\mathcal{T}, \mathcal{G}_+)$. As shown in [BHZ16] and although it may not look so at first sight, this construction is equivalent to the one given in [Hai14, Sec. 8] modulo a change of basis for $\mathcal{T}_+$.

Furthermore, given a collection $\{\xi_i^{(\varepsilon)}\}_{i=1}^m$ of smooth functions, we have a canonical lift to a model $(\Pi^{(\varepsilon)}, f^{(\varepsilon)})$ for $\mathscr{T}$ in the following way. First, we define a linear map $\mathbf{\Pi}^{(\varepsilon)} \colon \mathcal{T} \to \mathcal{C}^\infty$ by

$$\mathbf{\Pi}^{(\varepsilon)} \Xi_i = \xi_i^{(\varepsilon)} \,, \qquad \mathbf{\Pi}^{(\varepsilon)} \tau\bar{\tau} = (\mathbf{\Pi}^{(\varepsilon)}\tau) \cdot (\mathbf{\Pi}^{(\varepsilon)}\bar{\tau}) \,, \tag{2.6a}$$

$$\mathbf{\Pi}^{(\varepsilon)} X^k \tau = x^k \mathbf{\Pi}^{(\varepsilon)} \tau \,, \qquad \mathbf{\Pi}^{(\varepsilon)} \mathcal{I}_k \tau = D^k K \star \mathbf{\Pi}^{(\varepsilon)} \tau \,, \tag{2.6b}$$

where the kernel K is a truncated heat kernel as in [Hai14], $\star$ is convolution in $\mathbf{R}^2$, and x^k denotes the function $x \mapsto x^k$. In general, we define

Definition 2.2 A linear map $\mathbf{\Pi} \colon \mathcal{T} \to \mathcal{C}^\infty$ is *admissible* if (2.6b) holds.

Given any admissible linear map $\mathbf{\Pi} \colon \mathcal{T} \to \mathcal{C}^\infty$, there is a natural way of assigning to it a collection of characters $g_z^+(\mathbf{\Pi}) \colon \hat{\mathcal{T}}_+ \to \mathbf{R}$ by setting $g_z^+(\mathbf{\Pi}) X_i = z_i$, $g_z^+(\mathbf{\Pi}) \mathcal{I}_k(\tau) = (D^k K \star \mathbf{\Pi}\tau)(z)$, and then extending this multiplicatively. Write now $\hat{\mathcal{A}}_+ \colon \mathcal{T}_+ \to \hat{\mathcal{T}}_+$ for the unique algebra morphism such that $\hat{\mathcal{A}}_+ X_i = -X_i$ and

$$\hat{\mathcal{A}}_+ \mathcal{I}_k(\tau) = -\sum_{\ell} \frac{(-X)^\ell}{\ell!} \pi_+ \hat{\mathcal{M}}_+ (\mathcal{I}_{k+\ell} \otimes \hat{\mathcal{A}}_+) \Delta^+ \tau \,,$$

where this time the $\mathcal{I}_k$ are interpreted as maps $\mathcal{T} \to \hat{\mathcal{T}}_+$ and π_+ is the projection in $\hat{\mathcal{T}}_+$ onto the terms of positive homogeneity. (Similarly to above, $\hat{\mathcal{M}}_+$ is the product in $\hat{\mathcal{T}}_+$.) Comparing this to the definition of $\mathcal{A}_+$ given in (2.4), we see that it is

virtually identical, with the exception of the appearance of the projection π_+ and the fact that the two operators do not act on the same spaces.

Given $\mathbf{\Pi}$, we then define $\Pi_z : \mathcal{T} \to \mathcal{C}^\infty$ and $f_z \in \mathcal{G}_+$ by

$$f_z = g_z^+(\mathbf{\Pi})\hat{\mathcal{A}}_+ \,, \qquad \Pi_z = (\mathbf{\Pi} \otimes f_z)\Delta^+ \,, \tag{2.7}$$

for every $z \in \mathbf{R}^2$. We also define $\Gamma_{z\bar{z}} : \mathcal{T} \to \mathcal{T}$ by

$$\Gamma_{z\bar{z}} = (\mathrm{id} \otimes f_z \mathcal{A} \otimes f_{\bar{z}})(\Delta^+ \otimes \mathrm{id})\Delta^+ = (\gamma_{z\bar{z}} \otimes \mathrm{id})\Delta^+ \,, \tag{2.8}$$

with $\gamma_{z\bar{z}} = (f_z \mathcal{A} \otimes f_{\bar{z}})\Delta^+$, so that $\Pi_z \Gamma_{z\bar{z}} = \Pi_{\bar{z}}$ and we henceforth write $\mathfrak{Z}$ for the map

$$\mathfrak{Z} : \mathbf{\Pi} \mapsto (\Pi, \Gamma) \,,$$

given by (2.7) and (2.8). It is then possible to verify that if $\mathbf{\Pi}^{(\varepsilon)}$ is given by (2.6), then $\mathfrak{Z}(\mathbf{\Pi}^{(\varepsilon)})$ is indeed an admissible model for $\mathscr{T}$. For a generic admissible linear map $\mathbf{\Pi} : \mathcal{T} \to \mathcal{C}^\infty$ however this is not necessarily true.

Writing $\mathscr{M}$ for the space of admissible models for $\mathscr{T}$ that are periodic in the spatial variable, one can build a solution map $\mathcal{S} : \mathscr{M} \times \mathscr{L}^\alpha \mathscr{M} \to \bar{\mathcal{C}}(\mathbf{R}_+, \mathscr{L}^\alpha \mathscr{M})$ (here one should take $\alpha \in (0, \frac{1}{2} - \kappa)$, $\mathscr{L}^\alpha \mathscr{M}$ denotes the space of α-Hölder continuous loops in $\mathscr{M}$, and $\bar{\mathcal{C}}(\mathbf{R}_+, X)$ denotes the space of continuous functions with values in the metric space X, up to some explosion time at which they leave every bounded region of X) with the following two properties:

- If $\mathbf{\Pi}^{(\varepsilon)}$ is the canonical lift for some smooth functions $\xi_i^{(\varepsilon)}$, then $\mathcal{S}(\mathfrak{Z}(\mathbf{\Pi}^{(\varepsilon)}), u_0)$ is the maximal solution to (1.6) with initial condition u_0 and ξ_i replaced by ξ_i^ε.
- The map $\mathcal{S}$ is locally Lipschitz continuous in both of its arguments.

This shows that if it were the case that $\mathfrak{Z}(\mathbf{\Pi}^{(\varepsilon)})$ converges to some limiting model in $\mathscr{M}$ as $\varepsilon \to 0$, then Theorem 1.1 would follow at once. Unfortunately, this is simply not the case. We would therefore like to be able to "tweak" this model in such a way that it remains an admissible model but has a chance of converging as $\varepsilon \to 0$.

A natural way of "tweaking" $\mathbf{\Pi}^{(\varepsilon)}$ is to compose it with some linear map $M : \mathcal{T} \to \mathcal{T}$. This naturally leads to the following question: what are the linear maps M which are such that if $\mathfrak{Z}(\mathbf{\Pi})$ is an admissible model, then $\mathfrak{Z}(\mathbf{\Pi}M)$ is also an admissible model? More precisely, we give the following definition.

Definition 2.3 A linear map $M : \mathcal{T} \to \mathcal{T}$ is an *admissible renormalisation procedure* if, for every $\mathbf{\Pi} : \mathcal{T} \to \mathcal{C}^\infty$ such that $\mathfrak{Z}(\mathbf{\Pi}) \in \mathscr{M}$, one has $\mathfrak{Z}(\mathbf{\Pi}M) \in \mathscr{M}$.

In [Hai14, Sec. 8.3], a rather indirect characterisation of renormalisation procedures M is given. The aim of the remainder of this section is to give an explicit description of a very large class of such M. In order to describe these maps, we

recall that the elements of $\mathscr{W}$ can best be described by rooted trees with additional decorations in the following way. We have $m+1$ types of edges: m of them represent the symbols Ξ_i, and they always touch a leaf. The last type represents $\mathcal{I}$ and is decorated with an $\mathbf{N}^2$-valued label k, thus representing $\mathcal{I}_k$. Nodes are also decorated with $\mathbf{N}^2$-valued labels, representing factors of X^k. Finally, multiplication is represented by concatenation at the root. If we draw $\cdot$ for nodes, $\circ$ for Ξ_i (this looks like a node, but can unambiguously be interpreted as an edge protruding from the center of the disk since we postulated that these edges always terminate in a leaf), — for $\mathcal{I}$ and — for $\mathcal{I}'$, we do for example have

$$\mathcal{I}(\mathcal{I}'(\Xi_i)\mathcal{I}'(\Xi_j)) = \text{Y}\,, \qquad \mathcal{I}(\mathcal{I}'(\Xi_i)\mathcal{I}'(\Xi_j\mathcal{I}(\Xi_k))) = \text{Y}\,,$$

etc, with the understanding that $\circ$ denotes the relevant Ξ_i. The expression $\mathcal{I}'(X^k\mathcal{I}'(\Xi_i)\mathcal{I}'(\Xi_j))$ for example would be represented by Y, but with a decoration k at the center node where the three bold lines meet. This gives a bijection between canonical basis vectors of $\mathcal{F}$ and triples $(F, \mathfrak{n}, \mathfrak{e})$ where $F = (V_F, E_F)$ is a rooted trees with edge types in $\{\mathcal{I}, \Xi_1, \dots, \Xi_m\}$ subject to the restrictions described above, $\mathfrak{n}\colon V_F \to \mathbf{N}^2$, and $\mathfrak{e}\colon E_F \to \mathbf{N}^2$.

By analogy with the BPHZ renormalisation procedure [BP57, Hep69, Zim69], one would like to consider renormalisation maps that consist in "contracting subtrees". In order to formalise such an operation, consider a tree $T = (V_T, E_T)$, as well as a subforest $A = (V_A, E_A) \subset T$, i.e. an arbitrary subgraph of T which contains no isolated vertices. We then write $\mathcal{R}_A T$ for the tree obtained by contracting the connected components of A in T.

We also write $\hat{\mathcal{T}}_-$ for the free commutative algebra generated by $\mathscr{W}$ and $\mathcal{J}_+ \subset \hat{\mathcal{T}}_-$ for the ideal generated by $\{\tau \in \mathscr{W} \,:\, |\tau| \geq 0\}$. We interpret elements of $\hat{\mathcal{T}}_-$ as triples $(F, \mathfrak{n}, \mathfrak{e})$ as above, except that F is now allowed to be a forest. We also define the space $\mathcal{T}_-$ by $\mathcal{T}_- = \hat{\mathcal{T}}_-/\mathcal{J}_+$. With these notation at hand, we then define a map $\Delta^-\colon \mathcal{T} \to \mathcal{T}_- \otimes \mathcal{T}$ by setting, for $\tau = (T, \mathfrak{n}, \mathfrak{e}) \in \mathscr{W}$,

$$\Delta^-\tau = \sum_{A \subset T} \sum_{\mathfrak{e}_A, \mathfrak{n}_A} \frac{1}{\mathfrak{e}_A!} \binom{\mathfrak{n}}{\mathfrak{n}_A} (A, \mathfrak{n}_A + \pi\mathfrak{e}_A, \mathfrak{e}\restriction E_A) \otimes (\mathcal{R}_A F, \mathcal{R}_A(\mathfrak{n} - \mathfrak{n}_A), \mathfrak{e} + \mathfrak{e}_A)\,. \tag{2.9}$$

Here, the sum runs over all $\mathfrak{n}_A\colon V_A \to \mathbf{N}^2$ and $\mathfrak{e}_A\colon \partial(A, F) \to \mathbf{N}^2$, where $\partial(A, F)$ denotes the edges in $E_F \setminus E_A$ that are adjacent to V_A. Also, for a function $\mathfrak{m}\colon S \to \mathbf{Z}^2$ with S a finite set, we write $\mathfrak{m}! = \prod_{x \in S} \mathfrak{m}(x)_0!\mathfrak{m}(x)_1!$ and similarly for the binomial coefficients, with the convention that $k! = \infty$ for $k < 0$. As before, the sum appearing here is actually finite because all but finitely many terms have the first factor vanish in $\mathcal{T}_-$.

Our motivation for the definition of Δ^- is as follows. Assigning a number to each $\tau \in \mathscr{W}$ with $|\tau| < 0$ is equivalent to choosing an algebra morphism $g\colon \mathcal{T}_- \to \mathbf{R}$.

If we ignore for a moment the labels $\mathfrak{n}$ and $\mathfrak{e}$, an operation of the type $M_g\colon \mathcal{T} \to \mathcal{T}$ with

$$M_g\tau = (g \otimes \mathrm{id})\Delta^-\tau\ , \tag{2.10}$$

then corresponds to iterating over all ways of contracting subtrees of negative degree contained in τ and replacing them by the corresponding constant assigned to it by g. This corresponds to replacing a kernel of possibly several variables by a multiple of a Dirac delta function forcing all arguments to collapse. The seemingly complicated combinatorics appearing in (2.9) then encodes the possibility to also replace it by higher order derivatives of such a delta function in all of its arguments.

Similarly to before, Δ^- can also be viewed as a map $\Delta^-\colon \mathcal{T}_- \to \mathcal{T}_- \otimes \mathcal{T}_-$ by extending it multiplicatively from $\mathcal{W}$ to all of $\mathcal{T}_-$. Writing $\bullet$ for the product in $\mathcal{T}_-$ (which has nothing to do with the product we have on $\mathcal{W}$!), the following analogue to Proposition 2.1 was shown in [BHZ16].

Proposition 2.4 *There exists an algebra morphism $\mathcal{A}_-\colon \mathcal{T}_- \to \mathcal{T}_-$ so that $(\mathcal{T}_-, \bullet, \Delta^-, \mathcal{A}_-)$ is a Hopf algebra, and the map Δ^-, viewed as a map $\mathcal{T} \to \mathcal{T}_- \otimes \mathcal{T}$, turns $\mathcal{T}$ into a left comodule for $\mathcal{T}_-$.*

If we write just as before $\mathcal{G}_-$ for the group of characters $g\colon \mathcal{T}_- \to \mathbf{R}$, this yields a right action of $\mathcal{G}_-$ onto $\mathcal{T}$ by $g \mapsto M_g$ with M_g as in (2.10). The following is then the main result of [BHZ16].

Theorem 2.5 *For every $g \in \mathcal{G}_-$, the map M_g is an admissible renormalisation procedure.*

The idea of the proof of this theorem goes as follows. Assume for a moment that one can also find a map $\Delta^-\colon \hat{\mathcal{T}}_+ \to \hat{\mathcal{T}}_- \otimes \hat{\mathcal{T}}_+$ such that

$$\Delta^-\mathcal{I}_- \subset \mathcal{I}_+ \otimes \hat{\mathcal{T}}_+ + \hat{\mathcal{T}}_- \otimes \mathcal{I}_-\ ,$$

with $\mathcal{I}_\pm$ defined before (2.2) and (2.9). In particular, Δ^- passes through the quotients to a map $\mathcal{T}_+ \to \mathcal{T}_- \otimes \mathcal{T}_+$, which we assume to satisfy the following.

- On $\mathcal{T}$, one has the identity

$$\mathcal{M}_-(\Delta^- \otimes \Delta^-)\Delta^+ = (\mathrm{id} \otimes \Delta^+)\Delta^-\ , \tag{2.11}$$

  where

$$\mathcal{M}_-\colon \mathcal{T}_- \otimes \mathcal{T} \otimes \mathcal{T}_- \otimes \mathcal{T}_+ \to \mathcal{T}_- \otimes \mathcal{T} \otimes \mathcal{T}_+ \tag{2.12}$$

  is the map that multiplies the two factors in $\mathcal{T}_-$. The same is also true on $\mathcal{T}_+$.
- On $\mathcal{T}_+$, one has the identity

$$\Delta^-\hat{\mathcal{A}}_+ = (\mathrm{id} \otimes \hat{\mathcal{A}}_+)\Delta^-\ . \tag{2.13}$$

- The actions of $\mathcal{G}_-$ onto $\mathcal{T}$ and $\mathcal{T}_+$ given by (2.10) and the analogous formula for $\mathcal{T}_+$ only increase degrees.

In this case, it is straightforward to verify that, for any $g \in \mathcal{G}_-$, if we write M_g as before, set $\mathbf{\Pi}^g = \mathbf{\Pi} M_g$ for some $\mathbf{\Pi}$ such that $\mathfrak{Z}(\mathbf{\Pi}) = (\Pi, \Gamma)$ is a model, write $\mathfrak{Z}(\mathbf{\Pi}^g) = (\Pi^g, \Gamma^g)$, and define $\gamma_{z\bar{z}}$ and $\gamma^g_{z\bar{z}}$ as in (2.8), one has

$$\gamma^g_{z\bar{z}} = (g \otimes \gamma_{z\bar{z}})\Delta^- \;, \qquad \Pi^g_z = (g \otimes \Pi_z)\Delta^- \;.$$

To show this, one first uses (2.13) to show that $f^g_z = (g \otimes f_z)\Delta^-$, where f and f^g are defined from $\mathbf{\Pi}$ and $\mathbf{\Pi}^g$ as in (2.7). One then uses (2.11) (on $\mathcal{T}$) to show that the required identity for Π^g_z holds. Finally, one uses (2.11) on $\mathcal{T}_+$ to show that if one views M_g as acting on $\mathcal{T}_+$ via (2.10), then its action distributes over the product defined in (2.5) in the sense that $(M_g f) \circ (M_g \bar{f}) = M_g(f \circ \bar{f})$, which then implies the required identity for $\gamma^g_{z\bar{z}}$. The fact that the action of M_g increases degrees guarantees that $\mathfrak{Z}(\mathbf{\Pi}^g)$ is again a model, provided that $\mathfrak{Z}(\mathbf{\Pi})$ is.

The problem is that (2.11) actually fails in our situation. However, it turns out that it can still be rescued by the following construction. We look for a larger space $\mathcal{T}^{\mathrm{ex}}$, together with corresponding spaces $\hat{\mathcal{T}}^{\mathrm{ex}}_+$, $\hat{\mathcal{T}}^{\mathrm{ex}}_-$ and ideals $\mathcal{J}^{\mathrm{ex}}_- \subset \hat{\mathcal{T}}^{\mathrm{ex}}_+$ and $\mathcal{J}^{\mathrm{ex}}_+ \subset \hat{\mathcal{T}}^{\mathrm{ex}}_-$, all of them $\mathbf{R}$-graded, such that the following properties hold.

- There are analogous maps to Δ^+ and Δ^- acting on these "extended" spaces and such that all of the algebraic relations described above are satisfied, including (2.11) and (2.13). In particular, one has a map $\mathfrak{Z}^{\mathrm{ex}}$ turning linear maps $\mathcal{T}^{\mathrm{ex}} \to \mathcal{C}^\infty$ into candidate models on the regularity structure $(\mathcal{T}^{\mathrm{ex}}, \mathcal{G}^{\mathrm{ex}}_+)$ defined in formally the same way as above.
- There exists a projection $\pi^{\mathrm{ex}} \colon \mathcal{T}^{\mathrm{ex}} \to \mathcal{T}$ which is a right inverse for the inclusion $\mathcal{T} \hookrightarrow \mathcal{T}^{\mathrm{ex}}$ and is such that for any admissible $\mathbf{\Pi} \colon \mathcal{T} \to \mathcal{C}^\infty$, $\mathfrak{Z}(\mathbf{\Pi})$ is a model if and only if $\mathfrak{Z}^{\mathrm{ex}}(\mathbf{\Pi}\pi^{\mathrm{ex}})$ is a model.
- There exists an algebra morphism $\pi^{\mathrm{ex}}_- \colon \mathcal{T}^{\mathrm{ex}}_- \to \mathcal{T}_-$ such that, for every $g \in \mathcal{G}_-$, one has
$$\pi^{\mathrm{ex}} M^{\mathrm{ex}}_{g\pi^{\mathrm{ex}}_-} = M_g \pi^{\mathrm{ex}} \;. \tag{2.14}$$

Once we have constructed these larger spaces, the proof of Theorem 2.5 is rather straightforward. Fix $g \in \mathcal{G}_-$ and $\mathbf{\Pi}$ such that $\mathfrak{Z}(\mathbf{\Pi})$ is a model. Then, by the second property above, in order to show that $\mathfrak{Z}(\mathbf{\Pi} M_g)$ is a model, it suffices to show that $\mathfrak{Z}^{\mathrm{ex}}(\mathbf{\Pi} M_g \pi^{\mathrm{ex}})$ is a model. However, by (2.14), we have $\mathfrak{Z}^{\mathrm{ex}}(\mathbf{\Pi} M_g \pi^{\mathrm{ex}}) = \mathfrak{Z}^{\mathrm{ex}}(\mathbf{\Pi}\pi^{\mathrm{ex}} M^{\mathrm{ex}}_{g\pi^{\mathrm{ex}}_-})$ and, again by the second property, we already know that $\mathfrak{Z}^{\mathrm{ex}}(\mathbf{\Pi}\pi^{\mathrm{ex}})$ is a model. We conclude by the fact that $M^{\mathrm{ex}}_{g\pi^{\mathrm{ex}}_-}$ is an admissible renormalisation map thanks to the argument given above, using the properties (2.11) and (2.13) for the maps $\Delta^\pm$ defined on the extended spaces.

3 Analytical aspects

At this stage, we have built a rather large group $\mathcal{G}_-$ acting on our space of formal expressions $\mathcal{T}$ by admissible renormalisation procedures. Consider now regularised space-time noises $\xi_i^{(\varepsilon)}$ as in (1.7) and define $\mathbf{\Pi}^{(\varepsilon)}$ as their canonical lift, defined by (2.6). The following result is a particular instance of the main theorem of [CH16].

Theorem 3.1 *There exists a choice of (deterministic) elements $g^\varepsilon \in \mathcal{G}_-$ such that, setting $M^\varepsilon = M_{g^\varepsilon}$ as in (2.10), the sequence of models $\mathcal{L}(\mathbf{\Pi}^{(\varepsilon)} M^\varepsilon)$ converges to a limiting model in $\mathscr{M}$.*

Before we give an idea of the proof of this theorem, let us show how the element g^ε determining the suitable renormalisation procedure is constructed. It turns out that this construction is very similar to the construction of the elements $f_z \in \mathcal{G}_+$ constructed in (2.7) in order to correctly recenter the model, so that its behaviour around a given space-time point z matches its degree. This is maybe not surprising since one can also view the renormalisation as some kind of "recentering procedure" except that this time, instead of insisting that the *evaluation* of the model at a given location vanishes for basis vectors of positive homogeneity, we would like to impose that the *expectation* of the model vanishes for basis vectors of negative homogeneity.

Recall equation (2.9) defining Δ^-. It follows from this definition that the antipode $\mathcal{A}_-$ for the Hopf algebra $\mathcal{T}_-$ is defined for $\tau = (F, \mathfrak{n}, \mathfrak{e})$ by the recursion

$$\mathcal{A}_-\tau = -\sum_{\substack{A \subset T \\ A \neq \emptyset}} \sum_{\mathfrak{e}_A, \mathfrak{n}_A} \frac{1}{\mathfrak{e}_A!} \binom{\mathfrak{n}}{\mathfrak{n}_A} (A, \mathfrak{n}_A + \pi \mathfrak{e}_A, \mathfrak{e} \restriction E_A) \bullet \mathcal{A}_-(\mathcal{R}_A F, \mathcal{R}_A(\mathfrak{n} - \mathfrak{n}_A), \mathfrak{e} + \mathfrak{e}_A) \, ,$$

where $\bullet$ is the product in $\mathcal{T}_-$ as before. As above, the renormalisation procedure involves a twisted antipode. In order to define this, recall that $\mathcal{T}_- = \hat{\mathcal{T}}_- / \mathcal{J}_+$ and write $\pi_- \colon \hat{\mathcal{T}}_- \to \hat{\mathcal{T}}_-$ for the projection onto elements of strictly negative degree. Similarly to above, we then define $\hat{\mathcal{A}}_- \colon \mathcal{T}_- \to \hat{\mathcal{T}}_-$ inductively as being the unique algebra morphism so that, on elements of the type $(F, \mathfrak{n}, \mathfrak{e})$ with F a single tree (so that it is identified with an element of $\mathcal{W}$), one has

$$\hat{\mathcal{A}}_-(F, \mathfrak{n}, \mathfrak{e}) = -\sum_{\substack{A \subset T \\ A \neq \emptyset}} \sum_{\mathfrak{e}_A, \mathfrak{n}_A} \frac{1}{\mathfrak{e}_A!} \binom{\mathfrak{n}}{\mathfrak{n}_A} \pi_-\big((A, \mathfrak{n}_A + \pi \mathfrak{e}_A, \mathfrak{e} \restriction E_A) \bullet \hat{\mathcal{A}}_-(\mathcal{R}_A F, \mathcal{R}_A(\mathfrak{n} - \mathfrak{n}_A), \mathfrak{e} + \mathfrak{e}_A)\big) \, .$$

We furthermore note that any *random* linear map $\mathbf{\Pi} \colon \mathcal{T} \to \mathcal{C}^\infty$ with finite expectation gives rise to a character $g^-(\mathbf{\Pi})$ on $\hat{\mathcal{T}}_-$ by simply setting

$$g^-(\mathbf{\Pi})\tau = \mathbf{E}(\mathbf{\Pi}\tau)(0) \, , \qquad \tau \in \mathcal{W} \, ,$$

and then extending it multiplicatively. In this setting, at least for instances of $\mathbf{\Pi}$ satisfying a suitable kind of stationarity, we then claim that the "correct" choice of character g^ε appearing in Theorem 3.1 is given by

$$g^\varepsilon = g^-(\mathbf{\Pi}^{(\varepsilon)})\hat{\mathscr{A}}_- \,.$$

Comparing this and (2.10) to (2.7), we see that the renormalisation procedure required to make our models converge to a finite limit is indeed formally identical (modulo changing $+$ / evaluation into $-$ / expectation) to the recentering procedure discussed before.

Combining these constructions with the results of [Hai14, Sec. 10], one concludes that Theorem 3.1 essentially follows as soon as one has an estimate of the type

$$\mathbf{E}\langle (g^-(\mathbf{\Pi}^{(\varepsilon)})\hat{\mathscr{A}}_- \otimes \mathbf{\Pi}^{(\varepsilon)} \otimes g_z^+(\mathbf{\Pi}^{(\varepsilon)})\hat{\mathscr{A}}_+)\mathscr{M}_-(\Delta^- \otimes \Delta^-)\Delta^+\tau, \varphi_z^\lambda\rangle^2 \lesssim \lambda^{2|\tau|}\,, \quad (3.1)$$

for every $\tau \in \mathscr{W}$ with $|\tau| < 0$, where $|\tau|$ denotes the degree of τ as before, $\mathscr{M}_-$ is as in (2.12), $\langle \cdot, \cdot \rangle$ is the usual scalar product in $L^2(\mathbf{R}^2)$, and φ_z^λ is the translate and rescaling of a test function $\varphi \in \mathscr{C}_0^\infty$ with sufficiently many derivatives bounded by 1 and with support in the ball of radius 1 around the origin as in [Hai15, Def. 3.3].

From an algebraic perspective, the definitions of Δ^- and $\hat{\mathscr{A}}_-$ are very strongly reminiscent of the Hopf-algebraic formulation of Zimmermann's forest formula [Kre98], which was further explored in [CK00, CK01]. More precisely, our space $\hat{\mathscr{T}}_-$ is analogous to the space $\mathscr{A}$ in [Kre98], the quotiented space $\mathscr{T}_-$ is analogous to the space $\mathscr{A}/\sim$, etc., so (3.1) is really a form of BPHZ theorem.

The difference is threefold. First, our basic combinatorial structure is described by collections of trees rather than Feynman diagrams. These can then be interpreted as generating Feynman diagrams when taking expectations, by contracting leaves according to Wick's theorem. (Or the cumulant formula if one considers a $\mathbf{\Pi}^{(\varepsilon)}$ which is not obtained from the lift of a Gaussian process.) Second, the result in [CH16] applies to very large class of kernels K, provided that they exhibit the "correct" behaviour near the origin and, unlike most related results that can be found in the literature, it doesn't rely on the driving noise being Gaussian. Finally, and this is really the main difference, we see both "positive" and "negative" renormalisations appearing in (3.1), while the usual calculations performed in the context of QFT only involve negative renormalisation. The purpose of the latter is to ensure that we obtain finite quantities in the limit $\varepsilon \to 0$. The former on the other hand is crucial in order to obtain the correct power of λ in the right hand side of (3.1).

Let us explain the main ingredients appearing in the proof of (3.1). First, by translation invariance, one can set $z = 0$. Using Wick's formula, the left hand side of (3.1) can then be written, for some $N > 2$, as

$$I_\tau^{\lambda,\varepsilon} = \int \varphi^\lambda(z_1)\varphi^\lambda(z_2)\,\mathscr{K}_\tau^{(\varepsilon)}(z_1, \dots, z_N)\,dz_1 \cdots dz_N\,, \qquad z_i \in \mathbf{R}^2\,,$$

for some smooth kernel $\mathscr{K}_\tau^{(\varepsilon)}$ which, as $\varepsilon \to 0$, converges to a smooth limit $\mathscr{K}_\tau$ on the configuration space $C_N = \{(z_1, \dots, z_N) : z_i \neq z_j \, \forall \, i \neq j\}$, but exhibits quite singular behaviour on the "big diagonal" where two or more arguments coincide.

In order to estimate an integral of this type, we use the following construction reminiscent of the Fulton-MacPherson compactification of C_N and already used in [HS14, HQ15]. First, note that if $\Lambda_\lambda \in (\mathbf{R}^2)^N$ denotes the support of $z \mapsto \varphi^\lambda(z_1)\varphi^\lambda(z_2)$, $\mathfrak{T}$ is a countable index set, and $\{D_\sigma\}_{\sigma\in\mathfrak{T}}$ are a collection of bounded regions exhausting all of $(\mathbf{R}^2)^N$, an integral as above can trivially be estimated by

$$|I_\tau^{\lambda,\varepsilon}| \lesssim \lambda^{-6} \sum_{\sigma\in\mathfrak{T}} \mathbf{1}_{D_\sigma \cap \Lambda_\lambda \neq \emptyset} |D_\sigma| \sup_{z\in D_\sigma} |\mathscr{K}_\tau^{(\varepsilon)}(z)| \, , \tag{3.2}$$

where $|D|$ denotes the Lebesgue measure of D and the factor λ^{-6} comes from the parabolic rescaling of φ^λ. Such a bound is close to optimal if the regions D_σ can be chosen in such a way that the integrand $\mathscr{K}_\tau^{(\varepsilon)}$ does not vary much over them.

A good index set $\mathfrak{T}$ turns out to be given by the set of rooted binary trees $T = (V_T, E_T)$ with $N+1$ leaves endowed with an integer label $n(x)$ at each interior vertex $x \in V_T$. We now build a map $\mathcal{S} \colon C_N \to \mathfrak{T}$ such that the sets D_σ are then given by $D_\sigma = \mathcal{S}^{-1}(\sigma)$. Given $z \in C_N$, $\mathcal{S}(z)$ is built by simultaneously looking at a sequence $\mathscr{P}_k$ of partitions of $[N] = \{0, \dots, N\}$ and $(T_k, n_k) = (V_k, E_k, n_k)$ of labelled graphs (with labels in $\mathbf{Z}\cup\{+\infty\}$) as follows. We set $\mathscr{P}_0 = \{\{0\}, \dots, \{N\}\}$, $V_0 = [N]$, $E_0 = \emptyset$, and $n_0(x) = +\infty$ for $x \in V_0$. Then, given $\mathscr{P}_j$ and (T_j, n_j), we define the next element in the sequence as follows. If $\mathscr{P}_j = \{[N]\}$, then the construction stops and we set $\mathcal{S}(z) = (T_j, n_j)$. Otherwise, for every $A \in \mathscr{P}_j$, consider the set $z_A = \{z_i\}_{i\in A}$ (where we set $z_0 = 0$) and consider the pair (A, B) such that $d(z_A, z_B) \leq d(z_C, z_D)$ for every $C, D \in \mathscr{P}_j$, where d denotes the Hausdorff distance between compact sets. Since the points z_i are all distinct, the pair (A, B) is unique. One then sets

$$\begin{aligned} \mathscr{P}_{j+1} &= (\mathscr{P}_j \setminus \{A, B\}) \cup \{A \cup B\} \, , \\ V_{j+1} &= V_j \cup \{A \cup B\} \, , \quad E_{j+1} = E_j \cup \{(A \cup B, A), (A \cup B, B)\} \, . \end{aligned}$$

We furthermore define n_{j+1} to be equal to n_j on V_j and $n_{j+1}(A \cup B)$ to be the only integer such that

$$d(z_A, z_B) \leq 2^{-n_{j+1}(A\cup B)} < 2d(z_A, z_B) \, .$$

As a consequence of the properties of the Hausdorff distance, the labelled trees produced in this way have the property that if we partially order T in the natural way so that the root $[N]$ is minimal, it is always the case that $x \leq y$ implies $n(x) \leq n(y)$. The following lemma is crucial.

Lemma 3.2 *For $z \in C_N$, let $(T, n) = \mathcal{S}(z)$. Then, there exist constants c and C depending only on N such that, for any $i, j \in [N]$, and writing $i \wedge j = \sup\{x : x \leq \{i\} \,\&\, x \leq \{j\}\}$ for the most recent common ancestor of the leaves $\{i\}$ and $\{j\}$ in T, one has*

$$c2^{-n_{j+1}(i\wedge j)} \leq d(z_i, z_j) \leq C2^{-n_{j+1}(i\wedge j)} \,. \tag{3.3}$$

(Again with the convention $z_0 = 0$.) Furthermore, for $\sigma = (V, E, n)$, one has the upper bound

$$|D_\sigma| \lesssim \prod_{x\in V_\circ} 2^{-dn(x)} \,, \tag{3.4}$$

uniformly over $\sigma \in \mathfrak{T}$, where $V_\circ \subset V$ denotes the set of those vertices that are not leaves in T.

In other words, modulo constant factors, the distance between any two points of a given configuration z is completely determined by $\mathcal{S}(z)$. Furthermore, the tree structure provides a very efficient way of encoding the various constraints given by the triangle inequality. The aim then is to obtain, for any such binary tree $T = (V, E)$ a function $\bar\eta_T \colon V \to \mathbf{R}$, such that the integrand $\mathscr{K}_\tau^{(\varepsilon)}$ can be bounded by

$$|\mathscr{K}_\tau^{(\varepsilon)}(z)| \lesssim \prod_{x\in V_\circ} 2^{-\bar\eta_T(x)n(x)} \,, \tag{3.5}$$

uniformly in ε. Combining this with (3.4) and (3.3), one then obtains a bound of the type

$$|I_\tau^{\lambda,\varepsilon}| \lesssim \lambda^{-6} \sum_{T=(V,E)} \sum_n \prod_{x\in V_\circ} 2^{-\eta_T(x)\,n(x)} \,, \qquad \eta_T(x) = \bar\eta_T(x) + d \,, \tag{3.6}$$

where the inner sum runs over all weakly increasing maps $n \colon V_\circ \to \mathbf{Z}$ such that furthermore

$$2^{-n(0\wedge 1\wedge 2)} \leq C\lambda \,, \tag{3.7}$$

for some fixed $C > 1$. (Here $i \wedge j$ is as in Lemma 3.2.) This encodes the constraint that one only considers terms in the sum such that $D_\sigma \cap \Lambda_\lambda \neq \emptyset$. It is relatively straightforward to verify recursively that one has indeed a bound of the type (3.5) so that (3.6) holds with functions η_T satisfying $\sum_{x\in V_\circ} \eta_T(x) = |\tau| + 6$. It therefore remains to obtain conditions on such functions η_T guaranteeing that a sum of the type appearing in (3.6) is bounded by $C\lambda^{\sum_{x\in V_\circ}\eta_T(x)}$ for some fixed C.

This is the content of the following lemma.

Lemma 3.3 *Let $T = (V, E)$ be a rooted binary tree with leaves equal to $[N]$ with $N > 2$ and let $\eta \colon V_\circ \to \mathbf{R}$. Set $x_\star = 0 \wedge 1 \wedge 2 \in V_\circ$ and let $V_\star$ denote the nodes $x \in V_\circ$ lying on the path from $x_\star$ to the root, but not the root itself. Assume furthermore that the following two conditions hold.*

1. *For every* $x \in V_\circ$ *one has* $\sum_{y \geq x} \eta(y) > 0$.
2. *For every* $x \in V_\star$ *one has* $\sum_{y \not\geq x} \eta(y) < 0$.

Then, there exists a constant C *such that*

$$\sum_n \prod_{x \in V_\circ} 2^{\eta(x)\, n(x)} \leq C \lambda^{-\sum_{x \in V_\circ} \eta(x)}\,,$$

where the sum runs over all increasing $n \colon V_\circ \to \mathbf{Z}$ *satisfying* (3.7).

The proof of this lemma is relatively straightforward and can be found for example in [HQ15]. The two conditions appearing here are generalisations of the standard conditions on the integrability of a function with power law behaviour at the origin (corresponding to the first condition) and at infinity (corresponding to the second condition). The problem in our case is that if we simply replaced the complicated expression appearing in the left hand side of (3.1) by $\mathbf{E}\langle \mathbf{\Pi}^{(\varepsilon)} \tau, \varphi_z^\lambda \rangle^2$, then although (3.5) and (3.6) would still be satisfied for some η_T with $\sum_{x \in V_\circ} \eta_T(x) = |\tau| + 6$, both conditions of Lemma 3.3 would fail for a typical $\tau \in \mathscr{W}$.

The purpose of the two renormalisation procedures encoded by Δ^- and Δ^+ appearing in (3.1) is to allow us to obtain an improved bound which involves a function η_T that does satisfy the conditions of Lemma 3.3. In this procedure, the purpose of the "negative renormalisation" is precisely to ensure that the first condition is satisfied (thus removing small-scale divergences), while the purpose of the "positive renormalisation" is to ensure that the second condition is satisfied, guaranteeing integrability at large scales. The combinatorics of overlapping divergencies can in particular be unraveled by adapting tools from [FMRS85].

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