

Singular perturbations to semilinear stochastic heat equations

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Abstract

We consider a class of singular perturbations to the stochastic heat equation or semilinear variations thereof. The interesting feature of these perturbations is that, as the small parameter ε tends to zero, their solutions converge to the ‘wrong’ limit, i.e. they do not converge to the solution obtained by simply setting $\varepsilon = 0$. A similar effect is also observed for some (formally) small stochastic perturbations of a deterministic semilinear parabolic PDE.

Our proofs are based on a detailed analysis of the spatially rough component of the equations, combined with a judicious use of Gaussian concentration inequalities.

1 Introduction

A well-known ‘folklore’ statement is that if a problem (Φ) is well-posed, then the solutions to every ‘reasonable’ sequence (Φ_ε) of approximating problems should converge to it. This statement is of course backed by a huge number of rigorous mathematical statements, with virtually every theorem from numerical analysis being an example. As usual, the devil lies in the details, in this case the meaning of the word ‘reasonable’. However, if the problem Φ_0 obtained by formally setting $\varepsilon = 0$ in Φ_ε makes sense and is well-posed, then one does usually expect the solutions to Φ_ε to converge to that of Φ_0 .

One particularly famous exception to this rule is given by approximations to stochastic differential equations: if one constructs an approximation by smoothing out the driving noise, for example by piecewise linear interpolation on a scale ε , and then solving the corresponding random ODE, then the solutions converge to the Stratonovich interpretation of the original SDE [WZ65]. On the other hand, if one considers the usual Euler approximations with a step size of length ε , then these will converge to the Itô interpretation [KP92]. In this case however, one may argue that the original problem isn’t well-posed in the classical sense, since stochastic integrals are too irregular to be defined pathwise in an unambiguous way.

The aim of this article is to provide a class of examples of stochastic PDEs where a similar type of inconsistency occurs, but without having any ambiguity in the meaning of either the approximating or the limiting model. However, similar to the case of the approximations to an SDE, the inconsistency will be a consequence of the roughness of the solutions (but this time it is the spatial roughness that comes into play) and the correction term that appears in the limit can be interpreted as a kind of local quadratic variation, just as in the Itô-Stratonovich correction.

The aim of this article is to study the limit $\varepsilon \rightarrow 0$ of a class of stochastic PDEs of the type

$$\begin{aligned} \partial_t u_\varepsilon &= \nu \partial_x^2 u_\varepsilon + f(u_\varepsilon) + \sqrt{2} \xi(t, x) \\ &\quad - \varepsilon^2 \partial_x^4 u_\varepsilon + \varepsilon g(u_\varepsilon) \partial_x^2 u_\varepsilon + \varepsilon h(u_\varepsilon) (\partial_x u_\varepsilon \otimes \partial_x u_\varepsilon) \end{aligned} \quad (\Phi_\varepsilon)$$

where ξ is space-time white noise¹ and $x \in [0, 2\pi]$. For simplicity, we consider periodic boundary conditions, but we do not expect the result to depend on this. Note that additional constants in front of the noise term ξ can always be eliminated by a time change, whereas additional constants in front of the term $\partial_x^4 u$ can be eliminated by redefining ε , f , g and h . Here, we will assume that the solution u takes values in \mathbf{R}^n and that

$$f: \mathbf{R}^n \rightarrow \mathbf{R}^n, \quad g: \mathbf{R}^n \rightarrow L(\mathbf{R}^n, \mathbf{R}^n), \quad h: \mathbf{R}^n \rightarrow L(\mathbf{R}^n \otimes \mathbf{R}^n, \mathbf{R}^n),$$

are functions such that f and h are of class C^1 and g is C^2 . It is not too difficult to show [DPZ92, Hai09] that (Φ_ε) has unique local weak and mild solutions in L^∞ , so that our model is well-posed. Note furthermore that the stochastic noise term is additive, so that there is no need to resort to stochastic calculus to make sense of the solutions to (Φ_ε) . In particular, there is no ambiguity to the concept of solution to (Φ_ε) .²

Similarly, the model where we set $\varepsilon = 0$, namely

$$\partial_t u_0 = \nu \partial_x^2 u_0 + f(u_0) + \sqrt{2} \xi(t, x), \quad (\Phi_0)$$

is perfectly well-posed and has unique local solutions in L^∞ both in the mild and the weak sense.

It is therefore natural to expect that one has $u_\varepsilon \rightarrow u_0$ in some sense as $\varepsilon \rightarrow 0$. Surprisingly, this is not the case! Indeed, the purpose of this article is to prove a convergence result for (Φ_ε) that can loosely be formulated as:

Theorem 1.1 *The solutions u_ε to (Φ_ε) converge in probability as $\varepsilon \rightarrow 0$ to the solutions u to*

$$\partial_t u = \nu \partial_x^2 u + \bar{f}(u) + \sqrt{2} \xi(t, x), \quad (\Phi)$$

¹Recall that space-time white noise is the distribution-valued centred Gaussian process with covariance given by $\mathbf{E}\xi(x, t)\xi(y, s) = \delta(x - y)\delta(t - s)$.

²Since the solutions to (Φ_ε) are not C^2 in general (they are only C^α for every $\alpha < \frac{3}{2}$), the term $\varepsilon g(u_\varepsilon) \partial_x^2 u_\varepsilon$ has to be interpreted in the weak sense by performing one integration by parts.

where the effective nonlinearity \bar{f} is given by

$$\bar{f}(u) = f(u) + \frac{1}{2\sqrt{\nu}} \text{Tr}(h(u) - D_s g(u)) . \quad (1.1)$$

Furthermore, for every $\kappa > 0$, the convergence rate is of order $\mathcal{O}(\varepsilon^{\frac{1}{2}-\kappa})$ in the L^∞ -norm.

Remark 1.2 The partial trace of h is defined, in components, as $(\text{Tr } h)_i = \sum_j h_{ijj}$ and the symmetrised derivative of g is given by $(D_s g)_{ijk} = \frac{1}{2}(\partial_k g_{ij} + \partial_j g_{ik})$.

Remark 1.3 Note that no growth condition is assumed on f , g and h . Therefore, the limiting problem (Φ) could potentially have a finite explosion time τ^* . In this case, the approximation result obviously holds only up to τ^* . Furthermore, it requires to consider initial conditions that are ‘nice’ in the sense that they have the same structure as the solutions to (Φ) at positive times. A precise formulation of these conditions can be found in the rigorous version of the statement, Theorem 5.1 below.

Remark 1.4 The convergence rate $\varepsilon^{\frac{1}{2}-\kappa}$ is optimal in the sense that even in the ‘trivial’ linear case $f = g = h = 0$, the L^∞ -distance between u_ε and u is no better than $\varepsilon^{\frac{1}{2}}$ times some logarithmic correction factor.

In the case $g = 0$, one interpretation of our result is that the solutions to

$$\partial_t u_\varepsilon = \nu \partial_x^2 u_\varepsilon - \varepsilon^2 \partial_x^4 u_\varepsilon + f(u_\varepsilon) + \varepsilon h(u_\varepsilon) (\partial_x u_\varepsilon \diamond \partial_x u_\varepsilon) + \sqrt{2} \xi(t, x) ,$$

do indeed converge as $\varepsilon \rightarrow 0$ to the solutions to the corresponding system where we formally set $\varepsilon = 0$. In this equation, the Wick product \diamond should be interpreted as the Wick product with respect to the Gaussian structure induced on the solution space by the linearised equation, as for example in [JLM85, BG97, DPD03, DPT07]. Note that this notion of Wick product is *different* in general from the notion obtained by using the Gaussian structure induced by the driving noise (i.e. the Wick product of Wiener chaos) as in [HØUZ96, Cha00, MR10]. In our situation, we actually expect both notions of Wick product to lead to the same limit as $\varepsilon \rightarrow 0$, but we have no proof of this.

1.1 Extensions and related problems

There are a number of extensions to these results that are worth mentioning. Note first that if, instead of space-time white noise, we took any noise process with more regular samples (for example such that its covariance operator is given by $(-\partial_x^2)^{-\alpha}$ for some $\alpha > 0$), then the solutions to (Φ_ε) would indeed converge to those of (Φ_0) . However, if $\alpha \in (0, \frac{1}{2})$ and one replaces the factor ε in front of the nonlinearity in (Φ_ε) by $\varepsilon^{1-2\alpha}$, then one can retrace the proofs in this article to show that one has convergence to (Φ) , but with the factor $1/(2\sqrt{\nu})$ in (1.1) replaced by

$$\frac{1}{\pi \nu^{\alpha + \frac{1}{2}}} \int_0^\infty \frac{dx}{x^{2\alpha}(1+x^2)} .$$

Similarly, one could modify the linear part of the perturbation. Indeed, given a polynomial Q with $Q(0) = 1$ and positive leading coefficient, we could consider instead of (Φ_ε) the sequence of problems

$$\begin{aligned} \partial_t u_\varepsilon &= \nu Q(-\varepsilon^2 \partial_x^2) \partial_x^2 u_\varepsilon + f(u_\varepsilon) + \sqrt{2} \xi(t, x) \\ &\quad + \varepsilon g(u_\varepsilon) \partial_x^2 u_\varepsilon + \varepsilon h(u_\varepsilon) (\partial_x u_\varepsilon \otimes \partial_x u_\varepsilon) . \end{aligned}$$

Again, inspection of the proofs given in this article shows that one has convergence to (Φ) , but this time the prefactor of the correction term is given by

$$\frac{1}{\pi \nu} \int_0^\infty \frac{dx}{Q(x^2)} .$$

More interestingly, one can also consider situations where the limiting process is deterministic. For example, one can retrace the steps of the proof of Theorem 1.1 for the family of problems given by

$$\begin{aligned} \partial_t u_\varepsilon &= \nu \partial_x^2 u_\varepsilon + f(u_\varepsilon) + \sqrt{2} \varepsilon^\gamma \xi(t, x) \\ &\quad - \varepsilon^2 \partial_x^4 u_\varepsilon + \varepsilon^{1-2\gamma} g(u_\varepsilon) \partial_x^2 u_\varepsilon + \varepsilon^{1-2\gamma} h(u_\varepsilon) (\partial_x u_\varepsilon \otimes \partial_x u_\varepsilon) , \end{aligned}$$

and to show that for $\gamma \in (0, \frac{1}{2})$, the solutions converge to the solutions to the semilinear heat equation

$$\partial_t u = \nu \partial_x^2 u + \bar{f}(u) ,$$

with $\bar{f}(u)$ as in (1.1).

The borderline case $\gamma = \frac{1}{2}$ can also be treated, at least when $g = 0$, and we will present this case in detail since it appears to be more interesting and arises from a rather natural scaling. Consider the family of problems v_ε given by

$$\partial_t v_\varepsilon = \nu \partial_x^2 v_\varepsilon - \varepsilon^2 \partial_x^4 v_\varepsilon + f(v_\varepsilon) + h(v_\varepsilon) (\partial_x v_\varepsilon \otimes \partial_x v_\varepsilon) + \sqrt{2\varepsilon} \xi(t, x) . \quad (1.2)$$

We have the following result, which will be restated rigorously and proven in Section 5.1 below.

Theorem 1.5 *As $\varepsilon \rightarrow 0$, the solution v_ε to (1.2) converges to v , where*

$$\partial_t v = \nu \partial_x^2 v + \bar{f}(v) + h(v) (\partial_x v \otimes \partial_x v) .$$

Here, $\bar{f} = f + \frac{1}{2\sqrt{\nu}} \text{Tr } h$, and the convergence takes place in H^β for every $\beta \in (\frac{1}{2}, 1)$.

Finally, an even more interesting class of generalisations is obtained by removing the prefactor ε in front of the nonlinearity altogether, as in (1.2), but keeping a noise term of order 1 and subtracting the corresponding ‘infinite correction’ from the nonlinearity in the hope that a finite object is obtained in the limit. For example, one would like to be able to study the limit $\varepsilon \rightarrow 0$ in

$$\partial_t u_\varepsilon = \nu \partial_x^2 u_\varepsilon - \varepsilon^2 \partial_x^4 u_\varepsilon + h(u_\varepsilon) (\partial_x u_\varepsilon \otimes \partial_x u_\varepsilon) - \frac{\text{Tr } h(u_\varepsilon)}{2\sqrt{\nu\varepsilon}} + \sqrt{2} \xi(t, x) .$$

In the particular case $n = 1$ and $h = 1$, this reduces to the KPZ equation and it can be shown, using the Hopf-Cole transform, that a limit exists [BG97], at least for a slightly different kind of approximation. Unfortunately, no such tool exists in the general case and the problem of making sense of a KPZ-type equation directly is still wide open, although some results have been obtained in this direction [Cha00, Ass02, DPDT07].

1.2 Structure of the article

The remainder of the article is structured as follows. In Section 2, we show how equations of the type (Φ_ε) arise in a class of path sampling problems, thus giving an additional motivation to this work. This is followed by a brief heuristic calculation which explains why the limiting equation should have the nonlinearity \bar{f} instead of f and shows how to easily compute the prefactor $1/(2\sqrt{\nu})$ or the variations thereof that were discussed in the previous section. In Section 3, we give a number of preliminary calculations that are useful for the proof and that show how the solutions to these equations behave. Section 4 is devoted to the proof of the main technical bound that allows to control the difference between $\varepsilon(\partial_x u_\varepsilon \otimes \partial_x u_\varepsilon)$ and a suitable multiple of the identity matrix in some negative Sobolev norm. This bound is the main ingredient for the proof of Theorem 1.1 which is reformulated in a rigorous way and proved in Section 5.

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2 A motivation from path sampling

Besides pure mathematical interest and the, admittedly rather tenuous, connection to the KPZ equation discussed in the previous section, the study of this kind of singular limit is motivated by the following path sampling problem. Consider the ordinary differential equation describing gradient motion of a stochastic particle in a smooth potential V :

$$dq = -\nabla V(q) dt + \sqrt{2T} dW(t), \quad (2.1)$$

as well as the Langevin equation for a massive particle of mass m subject to the same potential:

$$dq = \dot{q} dt, \quad m d\dot{q} = -\nabla V(q) dt - \dot{q} dt + \sqrt{2T} dW(t). \quad (2.2)$$

(We consider this as a $2n$ -dimensional SDE for the pair (q, \dot{q}) .) It is well-known (see for example [Fre04, KPS04] for a recent exposition of this fact in the context of

a larger class of problems) that if one takes the limit $m \rightarrow 0$ in the second equation, one recovers the solution to (2.1). This is usually referred to as the Smoluchowski-Kramers approximation to (2.1) and can also be guessed in a ‘naive’ way by simply deleting the term $m d\dot{q}$ appearing in the left hand side of the second identity in (2.2) and noting that $\dot{q} dt = dq$ from the first identity.

Assume now that V has a critical point at the origin and consider a ‘bridge’ for both (2.1) and (2.2). In the case of (2.1), this is a solution starting at the origin and conditioned to return to the origin after a fixed time interval (which we assume to be equal to π in order to simplify some expressions in the sequel). In the case of (2.2), this is a sample from the stationary solution conditioned to pass through the origin at times 0 and π . Denote by μ_m the bridge for (2.2) and by μ_0 the bridge for (2.1) so that both μ_m and μ_0 are probability measures on the space $\mathcal{C}_0([0, \pi], \mathbf{R}^n)$ of continuous functions vanishing at their endpoints.

It was then shown in [HSVW05, HSV07] under some regularity and growth assumptions on V that μ_0 is the invariant measure for the stochastic PDE given by

$$du^i = \frac{1}{2T} \partial_t^2 u^i d\tau - \frac{1}{2T} \partial_{ij}^2 V(u) \partial_j V(u) d\tau + \frac{1}{2} \partial_{ijj}^3 V(u) d\tau + \sqrt{2} dW(\tau), \quad (2.3)$$

where W is a cylindrical process on $L^2([0, \pi], \mathbf{R}^n)$ (so that $\frac{dW}{dt}$ is space-time white noise) and the linear operator ∂_t^2 is endowed with Dirichlet boundary conditions. Summation over the index j is implied in those terms where it appears. Note that the variable t plays the role of *space* in this stochastic PDE, whereas the role of time is played by the ‘algorithmic time’ τ .

Actually, (2.3) is even reversible with respect to μ_0 and the corresponding Dirichlet form is given by

$$\mathcal{E}_0(\varphi, \varphi) = \int_{L^2([0, \pi], \mathbf{R}^n)} \|\nabla \varphi(u)\|^2 \mu_0(du), \quad (2.4)$$

where ∇ denotes the L^2 -gradient.

On the other hand, it was shown in [HSV09] that μ_m is the invariant measure for the fourth-order SPDE given by

$$\begin{aligned} du^i = & \frac{1}{2T} \partial_t^2 u^i d\tau - \frac{m^2}{2T} \partial_t^4 u^i d\tau - \frac{1}{2T} \partial_{ij}^2 V(u) \partial_j V(u) d\tau \\ & - \frac{m}{2T} \partial_{ij\ell}^3 V(u) \partial_x u^\ell \partial_x u^j d\tau - \frac{m}{T} \partial_{ij}^2 V(u) \partial_x^2 u^j d\tau + \sqrt{2} dW(\tau), \end{aligned} \quad (2.5)$$

where the linear operator is endowed with Dirichlet boundary conditions (for both u and its second derivative). Here, summation over repeated indices is also implied. The fact that 0 is a critical point for V was used in order to derive (2.5). If this was not the case, then the boundary conditions for the linear operator would be somewhat more involved, but the final result would be the same. Again, (2.5) is reversible with respect to μ_m and the corresponding Dirichlet form \mathcal{E}_m is given by (2.4) with μ_0 replaced by μ_m .

It is therefore a natural question to ask whether (2.3) can also be recovered as the limit of (2.5) when the mass m tends to zero. On the one hand, this should be the case since $\mathcal{E}_m(\varphi, \varphi) \rightarrow \mathcal{E}_0(\varphi, \varphi)$ for sufficiently regular test functions φ . On the other hand, taking the ‘naive’ limit $m \rightarrow 0$ in (2.5) by simply deleting all the terms that contain a factor m or m^2 yields

$$du^i = \frac{1}{2T} \partial_x^2 u^i d\tau - \frac{1}{2T} \partial_{ij}^2 V \partial_j V(u) d\tau + \sqrt{2} dW(\tau).$$

While this is indeed very similar in structure to (2.3), we note that the term containing third-order derivatives of V is missing!

Theorem 1.1 does indeed explain this apparent discrepancy since (2.5) is precisely of the form (Φ_ε) if we make the identifications

$$\varepsilon = \frac{m}{\sqrt{2T}}, \quad \nu = \frac{1}{2T},$$

and define the functions

$$f_i(u) = -\frac{\partial_{ij}^2 V(u) \partial_j V(u)}{2T}, \quad g_{ij}(u) = -\frac{2\partial_{ij}^2 V(u)}{\sqrt{2T}}, \quad h_{ij\ell}(u) = -\frac{\partial_{ij\ell}^3 V(u)}{\sqrt{2T}}.$$

We then have

$$\begin{aligned} \bar{f}_i(u) &= f_i(u) + \sqrt{\frac{T}{2}} (h_{ijj}(u) - \partial_j g_{ij}(u)) \\ &= -\frac{1}{2T} \partial_{ij}^2 V(u) \partial_j V(u) - \sqrt{\frac{T}{2}} \frac{\partial_{ijj}^3 V(u) - 2\partial_{ijj}^3 V(u)}{\sqrt{2T}} \\ &= -\frac{1}{2T} \partial_{ij}^2 V(u) \partial_j V(u) + \frac{1}{2} \partial_{ijj}^3 V(u). \end{aligned}$$

It therefore follows at once from Theorem 1.1 that (2.3) can indeed be recovered as the limit as $m \rightarrow 0$ of (2.5) as expected at the level of the corresponding Dirichlet forms.³

2.1 Heuristic explanation

Before we turn to the rigorous formulation of Theorem 1.1 and to its proof, let us give a formal argument why a careful analysis of (Φ_ε) reveals that \bar{f} should indeed have the form given in (1.1). We expand u_ε into Fourier modes $u_{\varepsilon,k} \in \mathbf{R}^d$ (we drop the subscript ε from now on and simply write u_k instead of $u_{\varepsilon,k}$), so that

$$u_\varepsilon(x, t) = \sum_{k \in \mathbf{Z}} u_k(t) e_k(x), \quad e_k(x) = \frac{\exp(ikx)}{\sqrt{2\pi}}. \quad (2.6)$$

³Strictly speaking, Theorem 1.1 does not apply to this situation since there we consider periodic rather than Dirichlet b.c.’s. We do not expect this to make any difference, but some of the arguments would be slightly more involved.

Since we only consider real-valued solutions, we furthermore have the constraint $u_{-k} = \bar{u}_k$. One would then expect the ‘high modes’ to behave roughly like the linear part of (Φ_ε) , that is for $k \gg 1$ one would expect to have $u_k(t) \approx \psi_k^\varepsilon(t)$ with

$$d\psi_k^{\varepsilon,i} = (-\nu k^2 - \varepsilon^2 k^4)\psi_k^{\varepsilon,i} dt + \sqrt{2} dW_k^i(t), \quad (2.7)$$

where the W_k^i 's are i.i.d. standard complex-valued Wiener processes. For fixed t , this means that $u_k^i(t) \approx \mathcal{N}(0, (\nu k^2 + \varepsilon^2 k^4)^{-1})$, which is the invariant measure for (2.7).

We treat the two terms $g(u)\partial_x^2 u$ and $h(u)(\partial_x u \otimes \partial_x u)$ appearing in (Φ_ε) separately, starting with the second term. The main contribution is expected to come from the ‘diagonal terms’ where no cancellations occur. Therefore, if these terms give a finite contributions to the limiting equation, one can reasonably expect the remaining terms to give a vanishing contribution. In order to compute the contribution of one of these terms, consider a product of the form $\varphi = vw^2$, where v is a function that has mainly low-frequency components and w has mainly high-frequency components that are furthermore independent and of zero mean. Since we have the identity

$$e_k(x)e_\ell(x)e_m(x) = \frac{1}{2\pi} e_{k+\ell+m}(x),$$

it follows that the components of φ are given by

$$\varphi_n = \frac{1}{2\pi} \sum_{k+\ell+m=n} v_m w_k w_\ell, \quad (2.8)$$

Since we assumed that the components w_k are independent and that v_m is ‘small’ for ‘large’ values of m , it is a reasonable expectation that the main contribution to this sum stems from the terms such that $k = -\ell$ (recall that the w_k are complex-valued random variables so that $\mathbf{E}w_k^2 = 0$, but $\mathbf{E}w_k w_{-k} = \mathbf{E}|w_k|^2 \neq 0$), so that

$$\varphi_n \approx \frac{v_n}{2\pi} \sum_{k \in \mathbf{Z}} |w_k|^2 = \frac{v_n}{2\pi} \|w\|^2.$$

Turning back to the term $h(u)(\partial_x u \otimes \partial_x u)$, these considerations and the fact that the high-frequency components of u_ε are expected to be close to those of ψ^ε suggest that one should have

$$\varepsilon h_{ijk}(u)(\partial_x u_j \partial_x u_k) \approx \varepsilon h_{ijj}(u)(\partial_x \psi^{\varepsilon,j})^2 \approx \delta_{kj} \frac{\varepsilon}{2\pi} h_{ijj}(u) \|\partial_x \psi^j\|^2.$$

On the other hand, one has

$$\begin{aligned} \frac{\varepsilon}{2\pi} \|\partial_x \psi^i\|^2 &\approx \frac{\varepsilon}{2\pi} \sum_{k \neq 0} \frac{k^2}{\nu k^2 + \varepsilon^2 k^4} \approx \frac{1}{2\pi} \sum_{k \in \mathbf{Z}} \frac{\varepsilon}{\nu + (\varepsilon k)^2} \\ &\approx \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{\nu + k^2} = \frac{1}{2\sqrt{\nu}}, \end{aligned}$$

so that one does indeed expect to have a contribution of

$$\varepsilon h(u)(\partial_x u \otimes \partial_x u) \approx \frac{1}{2\sqrt{\nu}} \operatorname{Tr} h(u) .$$

Let us now turn to the term containing g . If g was constant, then this term could be absorbed into the constant ν , thus not contributing to the limit. However, if g is not constant, then the high-frequency components of g can interact with the high-frequency components of $\partial_x^2 u$ in order to produce a non-vanishing contribution to the final result. If the high-frequency part of u is given by ψ , then the high-frequency part of $g(u)$ is in turn expected to be given by $Dg(u)\psi$, so that a reasoning similar to before yields one to expect

$$\varepsilon g(u)\partial_x^2 u \approx \varepsilon Dg(u)(\psi \otimes \partial_x^2 \psi) \approx \operatorname{Tr} Dg(u) \frac{\varepsilon}{2\pi} \sum_{|k| \gg 1} \frac{-k^2}{\nu k^2 + \varepsilon^2 k^4} \approx -\frac{\operatorname{Tr} Dg(u)}{2\sqrt{\nu}} .$$

One can also reach this conclusion by noting that $\varepsilon g(u)\partial_x^2 u = \varepsilon \partial_x (g(u)\partial_x u) - \varepsilon g'(u)(\partial_x u)^2$ and arguing that the first term should not matter because the first derivative gets ‘swallowed’ by the regularising properties of the Laplacian. This is actually the argument that will be used later on in the proof.

3 Preliminary calculations

Let us rewrite the solutions to (Φ) and (Φ_ε) as

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s)) ds + \psi^0(t) , \quad (3.1a)$$

$$u_\varepsilon(t) = S_\varepsilon(t)u_0 + \int_0^t S_\varepsilon(t-s)F_\varepsilon(u_\varepsilon(s)) ds + \psi^\varepsilon(t) , \quad (3.1b)$$

where we set

$$F_\varepsilon(u) = 1 + f(u) + \varepsilon g(u)\partial_x^2 u + \varepsilon h(u)(\partial_x u \otimes \partial_x u) , \quad F(u) = 1 + \bar{f}(u) . \quad (3.1c)$$

Here and throughout the remainder of this article, we denote by ψ^ε and ψ^0 the *stationary* solutions to the linearised equations

$$\begin{aligned} d\psi^\varepsilon(t) &= (\nu \partial_x^2 - 1)\psi^\varepsilon(t) dt - \varepsilon^2 \partial_x^4 \psi^\varepsilon(t) dt + \sqrt{2} dW(t) , \\ d\psi^0(t) &= (\nu \partial_x^2 - 1)\psi^0(t) dt + \sqrt{2} dW(t) . \end{aligned}$$

Here, W is a standard cylindrical Wiener process on $\mathcal{H} = L^2([0, 2\pi], \mathbf{R}^n)$ and the operators $\mathcal{L} = \partial_x^2$ and $\mathcal{L}_\varepsilon = \nu \partial_x^2 - \varepsilon^2 \partial_x^4$ are both endowed with periodic boundary conditions.

Remark 3.1 The reason why we subtract the constant 1 to the definitions of ψ^ε and ψ^0 is so that the 0-mode has also a stationary regime. This is also why we then add 1 to the definition of F to compensate.

We denote by $S(t)$ the semigroup generated by \mathcal{L} and by $S_\varepsilon(t)$ the semigroup generated by \mathcal{L}_ε , so that

$$\psi^0(t) = \sqrt{2} \int_{-\infty}^t S(t-s) dW(s),$$

and similarly for ψ^ε .

Remark 3.2 Note that u_0 is *not* the initial condition for either (3.1a) or (3.1b), since ψ^0 and ψ^ε are *stationary* solutions to the corresponding linear evolution equations. Therefore, one has $u(0) = u_0 + \psi^0(0)$ and $u_\varepsilon(0) = u_0 + \psi^\varepsilon(0)$.

We also denote by H^α with $\alpha \in \mathbf{R}$ the usual fractional Sobolev spaces, i.e. H^α is the domain of $(-\mathcal{L})^{\alpha/2}$, and we denote by $\|\cdot\|_\alpha$ the norm in H^α . It is straightforward to check that for every $\varepsilon > 0$, ψ^ε takes values in H^s for every $s < \frac{3}{2}$, while ψ^0 only takes values in H^s for $s < \frac{1}{2}$. We first note that one has the following local existence and uniqueness results [DPZ92, Hai09]:

Lemma 3.3 *Equation 3.1a admits a unique local mild solution for every $u_0 \in L^\infty$ and (3.1b) admits a unique local mild solution for every $u_0 \in H^1$.*

Proof. For 3.1a, it suffices to note that ψ^0 has continuous sample paths with values in L^∞ [DPZ92] and that F is locally Lipschitz from L^∞ into itself, so that a standard Picard fixed point argument applies. Similarly, ψ^ε has continuous sample paths with values in H^1 and it is easy to check that F_ε is locally Lipschitz from H^1 into H^{-1} . Since furthermore $\|S_\varepsilon(t)\|_{H^{-1} \rightarrow H^1} \leq C_\varepsilon/\sqrt{t}$, the claim follows again from a standard fixed point argument. \square

Remark 3.4 Note that at this stage we do not make any claim on the uniformity of bounds on the solutions to (3.1b) as $\varepsilon \rightarrow 0$.

In particular, this implies that, for every initial condition $u_0 \in L^\infty$, there exists a stopping time τ^* such that (3.1a) has a unique mild solution for $t < \tau^*$ and such that either $\tau^* = \infty$ or $\lim_{t \rightarrow \tau^*} \|u(t)\|_{L^\infty} = \infty$.

3.1 Semigroup

In this subsection, we give a few convenient bounds on the semigroups S and S_ε . Recall that for every $\alpha \geq \beta$, every $T > 0$ and every $\gamma > 0$, there exists a constant C such that

$$\|\exp(-(-\mathcal{L})^\gamma t)u\|_\alpha \leq Ct^{\frac{\beta-\alpha}{2\gamma}} \|u\|_\beta, \quad (3.2)$$

for every $u \in H^\beta$ and every $t \leq T$. This follows from standard semigroup theory and can also be checked directly by decomposing into Fourier modes.

Using this, we start by giving a sharp bound on the regularising properties of S_ε as a function of the parameter $\varepsilon > 0$. We have:

Lemma 3.5 For every $\alpha \geq \beta$ and $T > 0$, there exists $C > 0$ such that the bound

$$\|S_\varepsilon(t)u\|_\alpha \leq C(t^{\frac{\beta-\alpha}{2}} \wedge (\varepsilon^2 t)^{\frac{\beta-\alpha}{4}}) \|u\|_\beta,$$

holds for every $\varepsilon \leq 1$ and every $t \leq T$.

Proof. Since $\mathcal{L}_\varepsilon = \mathcal{L} - C\varepsilon^2 \mathcal{L}^2$ for some constant C , we have

$$\|S_\varepsilon(t)u\|_\alpha \leq C(\|S(t)u\|_\alpha \wedge \|\exp(-C\varepsilon^2 \mathcal{L}^2 t)u\|_\alpha),$$

so that the bound follows immediately from (3.2). \square

Lemma 3.6 For every $\gamma < 2$, every $\beta \leq \alpha + 2\gamma$, and every $T > 0$, there exists a constant C such that the bound

$$\|S(t)u - S_\varepsilon(t)u\|_\alpha \leq C\varepsilon^\gamma t^{\frac{\beta-\alpha-\gamma}{2}} \|u\|_\beta \quad (3.3)$$

holds for every $t \leq T$ and every $u \in L^2$.

Proof. Denote by u_k the k th Fourier mode of u , by r_k the k th Fourier mode of $r \stackrel{\text{def}}{=} S(t)u - S_\varepsilon(t)u$, and set $s_k(t) = e^{-(\nu k^2 + 1)t}$. We then have

$$|r_k| = s_k(t) |(e^{-\varepsilon^2 k^4 t} - 1)u_k| \leq s_k(t) (k^4 \varepsilon^2 t \wedge 1) |u_k| \leq C\varepsilon^\gamma k^{2\gamma} t^{\frac{\gamma}{2}} s_k(t) |u_k|.$$

It follows that $\|r\|_\alpha \leq C\varepsilon^\gamma t^{\frac{\gamma}{2}} \|S(t)u\|_{\alpha+2\gamma}$, so that the result follows again from (3.2). \square

Remark 3.7 Note that as operators from H^α to H^α , one has for every $T > 0$

$$\liminf_{\varepsilon \rightarrow 0} \sup_{t \in [0, T]} \|S(t) - S_\varepsilon(t)\|_\alpha > 0,$$

as can easily be seen by looking at the operators acting on e_k for suitably chosen k . This explains the presence of a divergence as $t \rightarrow 0$ in (3.3) if one takes $\alpha = \beta$ and $\gamma > 0$.

3.2 Stochastic convolution

We first recall the following quantitative version of Kolmogorov's continuity test:

Proposition 3.8 Let Ψ be a Gaussian random field on $[0, 1]^d$ with values in a separable Hilbert space \mathcal{H} such that there exist constants K , and $\alpha > 0$ such that

$$\mathbf{E} \|\Psi(x) - \Psi(y)\|^2 \leq K_2 |x - y|^\alpha,$$

holds for every x, y . Then, for every $p > 0$ there exist a constant C such that

$$\mathbf{E} \sup_{x \in [0, 1]^d} \|\Psi(x)\|^p \leq C |K_2 + \mathbf{E} \|\Psi(0)\|^2|^{\frac{p}{2}}.$$

Proof. It suffices to keep track of constants in the standard proof of Kolmogorov's criterion as in [RY91]. It also follows immediately from the Fernique-Talagrand result on the supremum of Gaussian processes [Fer74, Tal87]. \square

Denoting by $\psi_k^\varepsilon(t)$ the scalar product $\langle \psi^\varepsilon(t), e_k \rangle$, we then have

$$\mathbf{E}|\psi_k^\varepsilon|^2(t) = \frac{1}{1 + \nu k^2 + \varepsilon^2 k^4}.$$

It follows immediately that one has the following *a priori* bounds on ψ^ε :

Lemma 3.9 *There exist constants $C > 0$ and C_α such that the bounds*

$$\begin{aligned} \mathbf{E}\|\psi^\varepsilon(t)\|_\alpha^2 &\leq C_\alpha, & \alpha \in (-\infty, \tfrac{1}{2}), \\ \mathbf{E}\|\psi^\varepsilon(t)\|_\alpha^2 &\leq C|\log \varepsilon|, & \alpha = \tfrac{1}{2}, \\ \mathbf{E}\|\psi^\varepsilon(t)\|_\alpha^2 &\leq C_\alpha \varepsilon^{1-2\alpha}, & \alpha \in (\tfrac{1}{2}, \tfrac{3}{2}), \end{aligned}$$

hold. Furthermore, for every $\delta > 0$ and every $T > 0$, there exist constants \tilde{C}_α such that we have the bounds

$$\begin{aligned} \mathbf{E} \sup_{t \in [0, T]} \|\psi^\varepsilon(t)\|_\alpha^2 &\leq \tilde{C}_\alpha, & \alpha \in (-\infty, \tfrac{1}{2}), \\ \mathbf{E} \sup_{t \in [0, T]} \|\psi^\varepsilon(t)\|_\alpha^2 &\leq \tilde{C}_\alpha \varepsilon^{1-2\alpha-2\delta}, & \alpha \in [\tfrac{1}{2}, \tfrac{3}{2}). \end{aligned}$$

Proof. The first set of bounds follows from a straightforward explicit calculation. The second set of bounds follows similarly by applying Proposition 3.8. For example, for $\alpha \in [\frac{1}{2}, \frac{3}{2})$ and $\gamma \in (0, \frac{3}{4} - \frac{\alpha}{2})$, one obtains the bound

$$\begin{aligned} \mathbf{E}\|\psi^\varepsilon(t) - \psi^\varepsilon(s)\|_\alpha^2 &\leq \sum_{k \in \mathbf{Z}} \frac{k^{2\alpha}}{1 + \nu k^2 + \varepsilon^2 k^4} (1 \wedge |t - s|(\nu k^2 + \varepsilon^2 k^4)) \\ &\leq |t - s|^\gamma \sum_{k \in \mathbf{Z}} \frac{k^{2\alpha}}{(1 + \nu k^2 + \varepsilon^2 k^4)^{1-\gamma}} \\ &\approx \varepsilon^{1-2\alpha-2\gamma} |t - s|^\gamma \int_{-\infty}^{\infty} \frac{x^{2(\alpha+\gamma-1)}}{(\nu + x^2)^{1-\gamma}} dx, \end{aligned}$$

which implies the required bound by Proposition 3.8. The other bounds follow similarly. \square

Note that the function $\alpha \mapsto C_\alpha$ is bounded on any closed interval not including $\frac{1}{2}$ or $\frac{3}{2}$, but it diverges at these two values. Similarly, the function \tilde{C}_α diverges as it approaches $\frac{1}{2}$ from below. We now give a bound on the speed at which ψ^ε approaches ψ^0 as $\varepsilon \rightarrow 0$.

Proposition 3.10 *For every $\kappa > 0$ and every $T > 0$, there exists a constant C such that*

$$\mathbf{E} \sup_{t \in [0, T]} \|\psi^\varepsilon(t) - \psi^0(t)\|_{L^\infty} \leq C\varepsilon^{\frac{1}{2} - \kappa},$$

for every $\varepsilon < 1$.

Proof. Define the random fields

$$\Delta_\varepsilon^k(x, t) = (\psi_k^\varepsilon(t) - \psi_k^0(t)) e_k(x),$$

so that

$$\Delta_\varepsilon \stackrel{\text{def}}{=} \psi^\varepsilon - \psi^0 = \sum_{k \in \mathbf{Z}} \Delta_\varepsilon^k,$$

and the fields Δ_ε^k are all independent. Let us first obtain a bound on the second moment of $\Delta_\varepsilon^k(t, x)$. For $k = 0$, we have $\Delta_\varepsilon^0 = 0$, while for $k \neq 0$ we have the bound

$$\mathbf{E} |\psi_k^\varepsilon(0) - \psi_k^0(0)|^2 \leq \int_0^\infty e^{-2\nu k^2 t} (1 - e^{-\varepsilon^2 k^4 t})^2 dt.$$

Noting that there exists a constant C such that

$$(1 - e^{-\varepsilon^2 k^4 t})^2 \leq C(1 \wedge \varepsilon^4 k^8 t^2),$$

we can break this integral into two parts. For the first part, we obtain

$$\varepsilon^4 k^8 \int_0^{\varepsilon^{-2} k^{-4}} e^{-2\nu k^2 t} t^2 dt = \varepsilon^4 k^2 \int_0^{\varepsilon^{-2} k^{-2}} e^{-2\nu t} t^2 dt \leq C(\varepsilon^4 k^2 \wedge \varepsilon^{-2} k^{-4}).$$

For the second part, we obtain

$$\begin{aligned} \int_{\varepsilon^{-2} k^{-4}}^\infty e^{-2\nu k^2 t} dt &= k^{-2} \int_{\varepsilon^{-2} k^{-2}}^\infty e^{-2\nu t} dt \leq C k^{-2} \left(1 \wedge \int_{\varepsilon^{-2} k^{-2}}^\infty \frac{dt}{t^3}\right) \\ &= C(k^{-2} \wedge \varepsilon^4 k^2). \end{aligned}$$

Note now that if $k^{-2} < \varepsilon^4 k^2$, then one has $\varepsilon^{-2} k^{-4} < k^{-2}$ so that, combining these two bounds, we conclude that

$$\mathbf{E} |\psi_k^\varepsilon(0) - \psi_k^0(0)|^2 \leq C(k^{-2} \wedge \varepsilon^4 k^2). \quad (3.4)$$

In particular $\mathbf{E}(\Delta_\varepsilon^k(t, x))^2 \leq C(k^{-2} \wedge \varepsilon^4 k^2)$ by stationarity and the fact that the functions e_k are uniformly bounded.

On the other hand, $\psi_k^0(t)$ is a stationary Ornstein-Uhlenbeck process with both characteristic time and variance $1/(1 + \nu k^2)$. We thus have

$$\begin{aligned} \mathbf{E}(e_k(x)\psi_k^0(t) - e_k(y)\psi_k^0(s))^2 &\leq C\mathbf{E}|\psi_k^0(t) - \psi_k^0(s)|^2 + Ck^2|x - y|^2\mathbf{E}|\psi_k^0(s)|^2 \\ &\leq C|t - s| + C|x - y|^2, \end{aligned}$$

and it can easily be checked that the same bound also holds for $\psi_k^\varepsilon(t)$, uniformly in ε . It follows from this and (3.4) that

$$\begin{aligned} \mathbf{E}(\Delta_\varepsilon^k(x, t) - \Delta_\varepsilon^k(y, s))^2 &\leq C(k^{-2} \wedge \varepsilon^4 k^2 \wedge (|t - s| + |x - y|^2)) \\ &\leq C\varepsilon^{1-2\kappa} k^{-1-\kappa} (|t - s| + |x - y|^2)^{\frac{\kappa}{4}}. \end{aligned}$$

Since the Δ_ε^k are independent for different values of k , we then obtain

$$\begin{aligned} \mathbf{E}(\Delta_\varepsilon(x, t) - \Delta_\varepsilon(y, s))^2 &\leq \sum_{k \geq 1} \mathbf{E}(\Delta_\varepsilon^k(x, t) - \Delta_\varepsilon^k(y, s))^2 \\ &\leq C\varepsilon^{1-2\kappa} (|t - s| + |x - y|^2)^{\frac{\kappa}{4}}, \end{aligned}$$

so that the claim follows from Proposition 3.8. \square

4 Averaging results

The aim of this section is to show that, for any sufficiently regular function v , one has

$$v \cdot (\varepsilon \partial_x \psi^\varepsilon(t) \otimes \partial_x \psi^\varepsilon(t) - A) \rightarrow 0,$$

in a weak sense as $\varepsilon \rightarrow 0$, where A is a shorthand for $1/(2\sqrt{\nu})$ times the identity matrix. In fact, if $v \in H^\alpha$ for some $\alpha > \frac{1}{2}$ and the convergence is measured in a weak Sobolev space of sufficiently negative index (less than $-\frac{1}{2}$), then the convergence takes place at speed $\mathcal{O}(\sqrt{\varepsilon})$.

Let $\{w_k^\varepsilon\}_{k \in \mathbf{Z}}$ be a sequence of centred complex Gaussian random variables that are independent, except for the constraint $w_{-k} = \bar{w}_k$, and depending on a parameter $\varepsilon > 0$ with variance $\sigma_k = \frac{k^2}{1 + \nu k^2 + \varepsilon^2 k^4}$. We also set $\hat{\sigma}_k = \frac{1}{\nu + \varepsilon^2 k^2}$, and we note that one has

$$|\sigma_k - \hat{\sigma}_k| \leq \frac{C}{1 + k^2}. \quad (4.1)$$

One should think of the w_k as being the Fourier coefficients of one of the components of $\partial_x \psi^\varepsilon(t)$ for any fixed time t . We also consider $\{\tilde{w}_k^\varepsilon\}_{k \in \mathbf{Z}}$ a sequence with the same distribution but independent of the w_k 's. Think of \tilde{w} as describing a different component from the one that w describes.

Let furthermore v_k be a (possibly ε -dependent) sequence of complex random variables such that $v \in H^\alpha$ almost surely for some $\alpha > \frac{1}{2}$. With this notation, the n th Fourier modes of $\varphi = v(w^2 - 1/(2\varepsilon\sqrt{\nu}))$ and $\tilde{\varphi} = v w \tilde{w}$ are given by

$$\begin{aligned} \varphi_n &= \frac{1}{2\pi} \sum_{k+l+m=n} v_m w_k w_l - \frac{v_n}{2\varepsilon\sqrt{\nu}}, \\ \tilde{\varphi}_n &= \frac{1}{2\pi} \sum_{k+l+m=n} v_m w_k \tilde{w}_l. \end{aligned}$$

The aim of this section is to give sharp bounds on these quantities. Our main result in this setting is the following

Proposition 4.1 *Let v be a random variable such that $\|v\|_\alpha < \infty$ almost surely for some fixed $\alpha > \frac{1}{2}$. Then, for every $\gamma > \frac{1}{2}$, there exist constants C, c and δ such that the bound*

$$\mathbf{P}(\|\varphi\|_{-\gamma} \geq K) \leq \sum_{n \geq 1} \mathbf{P}(\|v\|_\alpha \geq Vn^\delta) + C \exp\left(-c \frac{\sqrt{\varepsilon} K}{V}\right), \quad (4.2)$$

hold for every $K > 0$ and every $V > 0$, and similarly for $\tilde{\varphi}$.

The main ingredient in the proof of Proposition 4.1 is the Gaussian concentration inequality for Lipschitz continuous functions [SC74, Bor75, Tal95], which we state here in the following form:

Proposition 4.2 *Let $G: \mathbf{R}^N \rightarrow \mathbf{R}$ be Lipschitz continuous with Lipschitz constant L and let X be a normal \mathbf{R}^N -valued Gaussian random variable. Then, there exists $c > 0$ independent of N such that the bound*

$$\mathbf{P}(|G(X) - \mathbf{E}G(X)| \geq K) \leq \exp\left(-c \frac{K^2}{L^2}\right), \quad (4.3)$$

holds for every $K > 0$.

Remark 4.3 The bound (4.3) also holds if X is centred Gaussian with a covariance matrix different from the identity, provided that the eigenvalues of the covariance matrix are bounded by some constant independent of N . In this case, the optimal constant c depends on that bound.

Another useful tool is the following little calculation:

Lemma 4.4 *For every $\delta > 0$ and every $\alpha < 1$, there exists a constant $C > 0$ such that the bound*

$$1 \wedge \sum_{n \geq 1} \exp(-Kn^\delta) \leq C \exp(-\alpha K),$$

holds for every $K > 0$.

Proof. The bound is trivial for $K < 1$, so that we assume $K \geq 1$ in the sequel. Since for every $\alpha < 1$ there exists a constant c such that $(1+x)^\delta \geq \alpha + cx^\delta$, we have

$$\begin{aligned} \sum_{n \geq 1} \exp(-Kn^\delta) &\leq e^{-K} + \int_1^\infty \exp(-Kx^\delta) dx = e^{-K} + \int_0^\infty \exp(-K(1+x)^\delta) dx \\ &\leq e^{-K} + e^{-\alpha K} \int_0^\infty \exp(-Kcx^\delta) dx = e^{-K} + CK^{-\frac{1}{\delta}} e^{-\alpha K}, \end{aligned}$$

as claimed. \square

We have now all the tools required for the

Proof of Proposition 4.1. We start with the bound on φ . Note first that since $\gamma > \frac{1}{2}$, there exists a constant C and some $\kappa > 0$ (any value less than $\gamma - \frac{1}{2}$ will do) such that the implication

$$\|\varphi\|_{-\gamma} \geq K \quad \Rightarrow \quad \exists n : |\varphi_n| \geq CKn^\kappa$$

holds for every $K > 0$. Furthermore, we see from the definition of φ that there exist elements $f^{(n)}$ such that φ_n can be written as $\varphi_n = \langle v, f^{(n)} \rangle$. In particular, $|\varphi_n| \leq \|v\|_\alpha \|f^{(n)}\|_{-\alpha}$, so that one has the implication

$$|\varphi_n| \geq CKn^\kappa \quad \Rightarrow \quad \|v\|_\alpha \geq Vn^\delta \quad \text{or} \quad \|f^{(n)}\|_{-\alpha} \geq C \frac{Kn^{\kappa-\delta}}{V}.$$

Combining these implications, we obtain the bound

$$\mathbf{P}(\|\varphi\|_{-\gamma} \geq K) \leq \sum_{n \geq 1} \mathbf{P}(\|v\|_\alpha \geq Vn^\delta) + \sum_{n \geq 1} \mathbf{P}\left(\|f^{(n)}\|_{-\alpha} \geq C \frac{Kn^{\kappa-\delta}}{V}\right). \quad (4.4)$$

Since the first term on the right hand side is exactly of the desired form, it remains to obtain suitable bounds on the $f^{(n)}$. Note at this point that if we want to have any chance to obtain the desired bound, we need to choose δ in such a way that $\kappa - \delta > 0$, which imposes the restriction $\delta < \gamma - \frac{1}{2}$.

We now break $f^{(n)}$ into a ‘diagonal’ part and a ‘off-diagonal’ part by writing $f^{(n)} = f^{(n,1)} + f^{(n,2)}$ with

$$f_m^{(n,1)} = \left(\frac{1}{2\pi} \sum_{k \in \mathbf{Z}} |w_k|^2 - \frac{1}{2\varepsilon\sqrt{\nu}} \right) \delta_{-n,m},$$

$$f_m^{(n,2)} = \frac{1}{2\pi} \sum'_{k+\ell=m+n} w_k w_\ell.$$

Here, the $'$ over the second sum indicates that we omit the terms with $k + \ell = 0$. The first term can be bounded in a straightforward manner. We have

$$\|f^{(n,1)}\|_{-\alpha} \leq \frac{1}{2\pi|n|^\alpha} \left| \frac{T_\nu}{\varepsilon} - \sum_{k \in \mathbf{Z}} |w_k|^2 \right|,$$

where we set $T_\nu = \frac{\pi}{\sqrt{\nu}}$ as a shorthand. At this stage, we note that $\sum_k \varepsilon \hat{\sigma}_k$ is nothing but a Riemann sum for the integral $\int_{-\infty}^{\infty} \frac{dx}{\nu+x^2}$. Since the value of this integral is precisely equal to T_ν and since the function $1/(\nu+x^2)$ is decreasing on $[0, \infty)$ and since $\sum_k |\sigma_k - \hat{\sigma}_k| \leq C$ by (4.1), we find that $|T_\nu - \sum_k \varepsilon \sigma_k| \leq C\varepsilon$ for some constant C . Applying Proposition 4.2 to the function $G(w) = \sqrt{\sum_{k \in \mathbf{Z}} |w_k|^2}$, it follows that the bound

$$\mathbf{P}(\|f^{(n,1)}\|_{-\alpha} \geq K) \leq C \exp(-c\sqrt{\varepsilon}Kn^\alpha), \quad (4.5)$$

holds for some constants c and C , uniformly in K .

We now aim for a similar bound for $f^{(n,2)}$. In order to do this, we make use of the following trick. Setting

$$f_n = \left(\varepsilon \left(\sum_{k \in \mathbf{Z}} |w_k|^2 \right)^2 + \sum_{m \in \mathbf{Z}} (1 + |m|)^{-2\alpha} \left| \sum'_{k+\ell=m+n} w_k w_\ell \right|^2 \right)^{1/4},$$

we have the almost sure bound $\|f^{(n,2)}\|_{-\alpha} \leq 2f_n^2$, so that it is sufficient to be able to get bounds on the f_n . Of course, this would also be true if the definition of f_n didn't include the term proportional to ε , but we will see shortly that this term is very useful in order to be able to apply Proposition 4.2.

We first obtain a bound on the expectation of f_n^4 :

$$\mathbf{E}|f_n|^4 \leq \varepsilon C \left(\mathbf{E} \sum_{k \in \mathbf{Z}} |w_k|^2 \right)^2 + \sum_{m \in \mathbf{Z}} (1 + |m|)^{-2\alpha} \mathbf{E} \left| \sum'_{k+\ell=m+n} w_k w_\ell \right|^2.$$

Since $\sum_k \sigma_k = \mathcal{O}(\varepsilon^{-1})$, the first term in this sum is of order ε^{-1} . Furthermore, since the w_k 's are independent, only the 'diagonal' terms remain when expanding the square under the second expectation. We thus have

$$\begin{aligned} \mathbf{E}|f_n|^4 &\leq \frac{C}{\varepsilon} + C \sum_{m \in \mathbf{Z}} (1 + |m|)^{-2\alpha} \left(\sum'_{k+\ell=m+n} \sigma_k \sigma_\ell \right) \\ &\leq \frac{C}{\varepsilon} + C \sum_{m \in \mathbf{Z}} (1 + |m|)^{-2\alpha} \sum_{k \in \mathbf{Z}} \sigma_k \leq \frac{C}{\varepsilon}, \end{aligned} \quad (4.6)$$

for some universal constant C , provided that $\alpha > \frac{1}{2}$. Here, we have used the fact that $\sum_k \sigma_k = \mathcal{O}(1/\varepsilon)$, as already mentioned previously. In particular, the bound (4.6) implies that $\mathbf{E}|f_n| \leq C\varepsilon^{-1/4}$ by Jensen's inequality.

Our next step is to obtain a bound on the Lipschitz constant of f_n as a function of the w 's. Denoting by ∂_j the partial derivative with respect to w_j , we have

$$2f_n^3 \partial_j f_n = \varepsilon w_{-j} \sum_{k \in \mathbf{Z}} |w_k|^2 + \sum_{m \in \mathbf{Z}} (1 + |m|)^{-2\alpha} w_{m+n-j} \sum'_{k+\ell=m+n} w_k w_\ell.$$

It then follows from the Cauchy-Schwartz inequality that

$$2|f_n^3 \partial_j f_n| \leq |f_n|^2 \left(\sqrt{\varepsilon} w_{-j} + \sqrt{\sum_{m \in \mathbf{Z}} (1 + |m|)^{-2\alpha} |w_{m+n-j}|^2} \right).$$

Setting $\|Df_n\|^2 = \sum_{j \geq 1} |\partial_j f_n|^2$ for the norm of the derivative of f_n , it follows that

$$\|Df_n\|^2 \leq C|f_n|^{-2} \left(\varepsilon \sum_{\ell \in \mathbf{Z}} |w_\ell|^2 + \sum_{k \in \mathbf{Z}} |w_k|^2 \sum_{m \in \mathbf{Z}} (1 + |m|)^{-2\alpha} \right). \quad (4.7)$$

Since $\alpha > \frac{1}{2}$ by assumption, the last sum in this term is finite. We see now why it was useful to add this additional term proportional to ε in the definition of f_n . This term indeed now allows us to obtain the bound

$$\sum_{k \in \mathbf{Z}} |w_k|^2 \leq \frac{1}{\sqrt{\varepsilon}} |f_n|^2,$$

(which would not hold otherwise!), thus yielding $\|Df_n\| \leq C\varepsilon^{-1/4}$. Combining this with (4.6), we conclude that the bound

$$\mathbf{P}(|f_n| \geq K) \leq \mathbf{P}(|f_n - \mathbf{E}f_n| \geq K - C\varepsilon^{-1/4}) \leq C \exp(-c\sqrt{\varepsilon}K^2),$$

holds by Remark 4.3, so that

$$\mathbf{P}(\|f^{(n,2)}\|_{-\alpha} \geq K) \leq C \exp(-c\sqrt{\varepsilon}K).$$

Combining this bound with (4.5), we conclude that

$$\mathbf{P}(\|f^{(n)}\|_{-\alpha} \geq K) \leq C \exp(-c\sqrt{\varepsilon}K), \quad (4.8)$$

for possibly different constants c and C . Inserting this into (4.4), we obtain

$$\mathbf{P}(\|\varphi\|_{-\gamma} \geq K) \leq \sum_{n \geq 1} \mathbf{P}(\|v\|_{\alpha} \geq Vn^{\delta}) + \sum_{n \geq 1} C \exp\left(-c\sqrt{\varepsilon} \frac{Kn^{\kappa-\delta}}{V}\right),$$

so that the desired bound follows from Lemma 4.4.

We now turn to the bound for $\tilde{\varphi}$, which is obtained in a very similar way. Similarly to before, we have $\tilde{\varphi}_n = \langle v, g^{(n)} \rangle$ for elements $g^{(n)}$ given by

$$g_m^{(n)} = \sum_{k+\ell=m+n} w_k \tilde{w}_{\ell}.$$

Similarly to before, we set

$$g_n^4 = \varepsilon \left(\sum_{m \in \mathbf{Z}} |w_m|^2 \right)^2 + \varepsilon \left(\sum_{m \in \mathbf{Z}} |\tilde{w}_m|^2 \right)^2 + \sum_{m \in \mathbf{Z}} (1 + |m|)^{-2\alpha} \left(\sum_{k+\ell=m+n} w_k \tilde{w}_{\ell} \right)^2,$$

so that the bound $\|g^{(n)}\|_{-\alpha} \leq |g_n|^2$ holds almost surely.

Let us first compute the expectation of $|g_n|^4$. Expanding the last square, we see that as previously, since w and \tilde{w} are independent, only the ‘diagonal’ terms contribute, so that

$$\mathbf{E}|g_n|^4 \leq \frac{C}{\varepsilon} + \sum_{m \in \mathbf{Z}} (1 + |m|)^{-2\alpha} \sum_{k+\ell=m+n} \sigma_k \sigma_{\ell} \leq \frac{C}{\varepsilon}.$$

Here, we have used the fact that $\sigma_k \leq 1/\nu$ to go from the first to the second line.

Since g_n is symmetric in w and \tilde{w} , it suffices to obtain a bound of the form $\|Dg_n\| \leq C\varepsilon^{-1/4}$ to conclude in the same way as before. We have the identity

$$2g_n^3 \partial_j g_n = \varepsilon w_{-j} \sum_{k \in \mathbf{Z}} |w_k|^2 + \sum_{m \in \mathbf{Z}} (1 + |m|)^{-2\alpha} \tilde{w}_{m+n-j} \left(\sum_{k+\ell=m+n} w_k \tilde{w}_{\ell} \right),$$

from which a bound similar to (4.7) (only with w replace by \tilde{w} in the second term) follows. The rest of the argument is identical to before. \square

We will mostly make use of the following two special cases. The first one is obvious from (4.2):

Corollary 4.5 *Let $\alpha, \gamma > \frac{1}{2}$ and let v be a random variable such that $\|v\|_\alpha \leq V$ almost surely. Then,*

$$\mathbf{P}(\|\varphi\|_{-\gamma} \geq K) \leq C \exp\left(-c \frac{\sqrt{\varepsilon} K}{V}\right),$$

and similarly for $\tilde{\varphi}$.

The second corollary does require a short calculation:

Corollary 4.6 *In the setting of Proposition 4.1, let \tilde{v}_n be a collection of independent centred Gaussian random variables such that $\mathbf{E}\tilde{v}_n^2 = \frac{\sigma_n}{n^\alpha}$ and assume that v is a random variable such that $\|v\|_\alpha \leq 1 + \|\tilde{v}\|_\alpha$ almost surely. Then,*

$$\mathbf{P}(\|\varphi\|_{-\gamma} \geq K) \leq C \exp\left(-c\varepsilon^{2\alpha/3} K^{2/3}\right), \quad (4.9)$$

and similarly for $\tilde{\varphi}$.

Proof. A straightforward calculation shows that

$$\mathbf{E}\|\tilde{v}\|_\alpha^2 \leq \sum_{n \geq 1} \frac{1}{n^{2-2\alpha}(\nu + \varepsilon^2 n^2)} = \mathcal{O}(\varepsilon^{1-2\alpha}),$$

so that by Fernique's theorem there exists a constant c such that

$$\mathbf{P}(\|\tilde{v}\|_\alpha > K) \leq \exp(-c\varepsilon^{2\alpha-1} K^2),$$

and similarly for v . It therefore follows from (4.2) and Lemma 4.4 that

$$\mathbf{P}(\|\varphi\|_{-\gamma} \geq K) \leq C \exp(-c\varepsilon^{2\alpha-1} V^2) + C \exp\left(-c\sqrt{\varepsilon} \frac{K}{V}\right).$$

Setting $V = \varepsilon^{\frac{1}{2} - \frac{2\alpha}{3}} K^{\frac{1}{3}}$ completes the proof. \square

Remark 4.7 Here we did not make any assumption regarding the correlations between the v_n and the w_n . In the special cases when either $v_n = \frac{w_n}{n}$ or v is independent of w and \tilde{w} , it is possible to check by an explicit calculation that the boundary $\alpha = \frac{1}{2}$ can be reached in (4.9). Since this is of no particular use to us, there is no need to make this additional effort.

5 Proof of the main result

This section is devoted to the proofs of Theorems 1.1 and 1.5. We first reformulate Theorem 1.1 in a more precise way:

Theorem 5.1 *Let u_0 be an $H^{\frac{3}{2}}$ -valued random variable. Then, for every $\kappa > 0$ and every $T > 0$, there exists a sequence of stopping times τ_ε with $\tau_\varepsilon \rightarrow T \wedge \tau^*$ in probability, so that*

$$\lim_{\varepsilon \rightarrow 0} \mathbf{P} \left(\sup_{t \in [0, \tau_\varepsilon]} \|u(t) - u_\varepsilon(t)\|_{L^\infty} > \varepsilon^{\frac{1}{2} - \kappa} \right) = 0 .$$

Remark 5.2 Assuming that u_0 takes values in $H^{\frac{3}{2}}$ may look unnatural at first sight since the solutions to (Φ) do not take values in that space. Note however that u_0 is *not* the initial condition to (Φ) since we have assumed that ψ^0 is a stationary realisation of the solution to the linearised problem. Therefore, the initial condition for the Cauchy problem (Φ) is given by $u_0 + \psi^0(0)$ and it turns out that the solution to (Φ) with continuous initial condition is of this form after any positive time. (Actually, it is even of this form with $u_0 \in H^s$ for every $s < \frac{5}{2}$.)

Remark 5.3 If the limiting problem has global solutions, then one has $\tau_\varepsilon \rightarrow T$ in probability. However, this does *not* imply that the approximating problem has global solutions! It only implies that its explosion time becomes arbitrarily large as $\varepsilon \rightarrow 0$. This is also why it would be unrealistic to expect approximation results in expectation rather than in probability.

In order to prove Theorem 5.1, we consider the following two intermediary problems:

$$\begin{aligned} u_\varepsilon^{(1)}(t) &= S(t)u_0 + \int_0^t S(t-s)F(u_\varepsilon^{(1)}(s)) ds + \psi^\varepsilon(t) , \\ u_\varepsilon^{(2)}(t) &= S_\varepsilon(t)u_0 + \int_0^t S_\varepsilon(t-s)F(u_\varepsilon^{(2)}(s)) ds + \psi^\varepsilon(t) , \end{aligned}$$

where the notations are the same as in (3.1). We are then going to bound first the L^∞ -norm of $u_\varepsilon^{(1)} - u$, then the H^α -norm of $u_\varepsilon^{(1)} - u_\varepsilon^{(2)}$ for some $\alpha > \frac{1}{2}$, and finally the H^1 -norm of $u_\varepsilon^{(2)} - u_\varepsilon$. Combining these bounds and using the Sobolev embedding $H^\alpha \hookrightarrow L^\infty$ then shows the required claim. We now proceed to obtain these bounds. In the sequel, one should think of the terminal time T , the exponent κ , and the (possibly random) ‘initial condition’ $u_0 \in H^{\frac{3}{2}}$ as being fixed once and for all.

We also consider a (large) cutoff value $K > 0$. One should think of K as being a fixed constant in the sequel although at the end we will take $K \rightarrow \infty$ as $\varepsilon \rightarrow 0$.

Notation 5.4 From now on, we will denote by C_K a generic constant that is allowed to depend only on the particulars of the problem at hand (f, g, h , etc), on the choice

of the cutoff value K , and possibly on κ . It is not allowed to depend on ε or on the initial condition u_0 . The precise value of C_K is allowed to change without warning, even from one line to the next within the same equation.

We define a first stopping time τ^K by

$$\tau^K = \inf\{s \leq T : \|u(s)\|_{L^\infty} \geq K\},$$

with the convention that $\tau^K = T$ if the set is empty. Note that $\tau^K \rightarrow T \wedge \tau^*$ in probability as $K \rightarrow \infty$ by the definition of τ^* .

We then set

$$\tau_1^K = \tau^K \wedge \inf\{s \leq T : \|u(s) - u_\varepsilon^{(1)}(s)\|_{L^\infty} \geq K\}.$$

Our first step is to show that before the time τ_1^K , the quantity $\|u(s) - u_\varepsilon^{(1)}(s)\|_{L^\infty}$ is small with very high probability:

Proposition 5.5 *For every $K > 0$, one has*

$$\lim_{\varepsilon \rightarrow 0} \mathbf{P}\left(\sup_{t \leq \tau_1^K} \|u(t) - u_\varepsilon^{(1)}(t)\|_{L^\infty} > \varepsilon^{\frac{1}{2}-\kappa}\right) = 0.$$

Proof. Define the ‘residue’ $R^{(1)}(t) = \psi^\varepsilon(t) - \psi^0(t)$ and set $\varrho(t) = u(t) - u_\varepsilon^{(1)}(t) + R^{(1)}(t)$, so that ϱ satisfies the integral equation

$$\varrho(t) = \int_0^t S(t-s)(F(u(s)) - F(u(s) - \varrho(s) + R^{(1)}(s))) ds.$$

Note now that for $s \leq \tau_1^K$, the two arguments of F are bounded by $2K$ in the L^∞ -norm by construction. Combining this with the local Lipschitz property of \bar{f} and the fact that S is a contraction semigroup in L^∞ , there exists a constant C_K such that

$$\|\varrho(t)\|_{L^\infty} \leq C_K \int_0^t (\|\varrho(s)\|_{L^\infty} + \|R^{(1)}(s)\|_{L^\infty}) ds.$$

The claim now follows at once from Proposition 3.10, combined with Gronwall’s inequality. \square

In our next step, we obtain a bound on the difference between $u_\varepsilon^{(1)}$ and $u_\varepsilon^{(2)}$ in a very similar way. We define as before the stopping time

$$\tau_2^K = \tau_1^K \wedge \inf\{s \leq T : \|u_\varepsilon^{(1)}(s) - u_\varepsilon^{(2)}(s)\|_\alpha + \|\psi^\varepsilon(s)\|_{L^\infty} \geq K\},$$

but this time it additionally depends on a parameter α to be determined later. One should think of α as being very close to, but slightly greater than, $1/2$. We then have,

Proposition 5.6 For every $\alpha \in (\frac{1}{2}, 1)$, one has

$$\lim_{\varepsilon \rightarrow 0} \mathbf{P} \left(\sup_{t \leq \tau_2^K} \|u_\varepsilon^{(1)}(t) - u_\varepsilon^{(2)}(t)\|_\alpha > \varepsilon^{\frac{1}{2} - \kappa} \right) = 0 .$$

Proof. Writing $v_\varepsilon^{(i)}(t) = u_\varepsilon^{(i)}(t) - \psi^\varepsilon(t)$, we have $v_\varepsilon^{(i)}(0) = u_0$ and

$$\begin{aligned} v_\varepsilon^{(1)}(t) &= S(t-s)v_\varepsilon^{(1)}(s) + \int_s^t S(t-r)F(v_\varepsilon^{(1)}(r) + \psi^\varepsilon(r)) dr , \\ v_\varepsilon^{(2)}(t) &= S_\varepsilon(t-s)v_\varepsilon^{(2)}(s) + \int_s^t S_\varepsilon(t-r)F(v_\varepsilon^{(2)}(r) + \psi^\varepsilon(r)) dr , \end{aligned}$$

As before, we define the residue $R^{(2)}(t) = S_\varepsilon(t)u_0 - S(t)u_0$ and we note that

$$\sup_{s \leq T} \|R^{(2)}(s)\|_\alpha \leq C\varepsilon^{\frac{1}{2}} \|u_0\|_{\alpha + \frac{1}{2}} ,$$

follows immediately from Lemma 3.6. Using the triangle inequality and the bound on the differences between the two semigroups given by Lemma 3.6, we furthermore obtain for any u, v with $\|u\|_{L^\infty} + \|v\|_{L^\infty} \leq K$ the bound

$$\begin{aligned} \|S(t)F(u) - S_\varepsilon(t)F(v)\|_\alpha &\leq \|(S(t) - S_\varepsilon(t))F(u)\|_\alpha + \|S_\varepsilon(t)(F(u) - F(v))\|_\alpha \\ &\leq C_K \varepsilon^{\frac{1}{2}} t^{-\frac{\alpha}{2} - \frac{1}{4}} + \bar{C}_K t^{-\frac{\alpha}{2}} \|u - v\|_{L^\infty} . \end{aligned}$$

In this bound, we single out the constant \bar{C}_K in order to be able to reuse its precise value later in the proof. Setting $\varrho = u_\varepsilon^{(1)} - u_\varepsilon^{(2)} = v_\varepsilon^{(1)} - v_\varepsilon^{(2)}$ we conclude that, for any $t \leq \tau_2^K$, we have

$$\|\varrho(t)\|_\alpha \leq C_K \varepsilon^{\frac{1}{2}} (1 + \|u_0\|_{\frac{3}{2}}) + \bar{C}_K t^{1 - \frac{\alpha}{2}} \sup_{0 < r < t} \|\varrho(r)\|_\alpha . \quad (5.1)$$

On the other hand, we obtain in a similar way from Lemma 3.6 the bound

$$\|S(t)u - S_\varepsilon(t)v\|_\alpha \leq \|u - v\|_\alpha + Ct^{-\frac{\alpha}{2} - \frac{1}{4}} \|u - v\|_{L^2} ,$$

so that

$$\|\varrho(t)\|_\alpha \leq \|\varrho(s)\|_\alpha + C_K \varepsilon^{\frac{1}{2}} (1 + (t-s)^{-\frac{\alpha}{2} - \frac{1}{4}}) + \bar{C}_K (t-s)^{1 - \frac{\alpha}{2}} \sup_{s < r < t} \|\varrho(r)\|_\alpha . \quad (5.2)$$

In order to obtain a bound on ϱ over the whole time interval $[0, \tau_2^K]$, we now fix some δ_K sufficiently small so that $\bar{C}_K (2\delta_K)^{1 - \frac{\alpha}{2}} \leq \frac{1}{2}$. Setting

$$r_k \stackrel{\text{def}}{=} \sup_{k\delta_K \wedge \tau_2^K \leq t \leq (k+1)\delta_K \wedge \tau_2^K} \|\varrho(t)\|_\alpha ,$$

it then follows from (5.1) that

$$r_0 \vee r_1 \leq C_K \varepsilon^{\frac{1}{2}} (1 + \|u_0\|_{\frac{3}{2}}) . \quad (5.3)$$

Then, from (5.2), we obtain

$$r_{k+1} \leq \|\varrho(k\delta_K \wedge \tau_2^K)\|_\alpha + C_K \varepsilon^{\frac{1}{2}} (1 + \delta_K^{-\frac{\alpha}{2} - \frac{1}{4}}) + \frac{1}{2}(r_k + r_{k+1}),$$

so that, since δ_K is independent of ε ,

$$r_{k+1} \leq 3r_k + C_K \varepsilon^{\frac{1}{2}}. \quad (5.4)$$

Combining the bounds (5.3) and (5.4) shows that $\sup_{t \leq \tau_2^K} \|\varrho(t)\|_\alpha \leq C_K \varepsilon^{1/2}$, thus concluding the proof. \square

It now remains to obtain a bound on the difference between $u_\varepsilon^{(2)}(t)$ and u_ε . This time, we are going to consider the H^1 norm, so that we set

$$\tau_3^K = \tau_1^K \wedge \inf\{s \leq T : \|u_\varepsilon^{(2)}(s) - u_\varepsilon(s)\|_1 \geq K\}.$$

Note first that one has the following a priori bound:

Lemma 5.7 *The bound*

$$\|u_\varepsilon(t) - \psi^\varepsilon(t)\|_1 \leq C_K$$

holds for every $t \leq \tau_3^K$.

Proof. It follows from the definition of $u_\varepsilon^{(2)}$ that

$$\|u_\varepsilon^{(2)}(t) - \psi^\varepsilon(t)\|_1 \leq \|u_0\|_1 + \int_0^t \|S_\varepsilon(t-s)F(u_\varepsilon^{(2)}(s))\|_1 ds.$$

Since $\|F(u_\varepsilon^{(2)}(s))\|_{L^\infty} \leq C_K$ for $t \leq \tau_3^K$, it follows from the regularising properties of the semigroup S_ε that

$$\|u_\varepsilon^{(2)}(t) - \psi^\varepsilon(t)\|_1 \leq C_K.$$

The claim then follows from the definitions of the various stopping times. \square

Proposition 5.8 *For every $\kappa > 0$ and every $K > 0$, one has*

$$\lim_{\varepsilon \rightarrow 0} \mathbf{P}\left(\sup_{t \leq \tau_3^K} \|u_\varepsilon^{(2)}(t) - u_\varepsilon(t)\|_1 > \varepsilon^{\frac{1}{2} - \kappa}\right) = 0.$$

Proof. As before, we write $\varrho = u_\varepsilon - u_\varepsilon^{(2)}$, so that

$$\varrho(t) = \int_0^t S_\varepsilon(t-s)(F_\varepsilon(u_\varepsilon(s)) - F(u_\varepsilon(s) - \varrho(s))) ds.$$

Note now that since $\|u_\varepsilon\|_{L^\infty}$ is bounded by a fixed constant before time τ_3^K , $\|F(u_\varepsilon) - F(u_\varepsilon - \varrho)\|_{L^\infty}$ is bounded by a constant (depending only on K) multiple of $\|\varrho\|_{L^\infty}$ so that, for any two times $s < t \leq \tau_3^K$, we have the almost sure bound

$$\|\varrho(t)\|_1 \leq \|\varrho(s)\|_1 + \left\| \int_s^t S_\varepsilon(t-r)(F_\varepsilon(u_\varepsilon(r)) - F(u_\varepsilon(r))) dr \right\|_1$$

$$+ C_K(t-s)^{\frac{1}{2}} \sup_{s < r < t} \|\varrho(r)\|_1 .$$

By breaking the interval $[0, T]$ into a finite number of subintervals $[t_k, t_{k+1}]$ of sufficiently short length, we conclude as above that

$$\sup_{t < \tau_3^K} \|\varrho(t)\|_1 \leq C_K \sup_k \sup_{t \in [t_k, t_{k+1}]} \left\| \int_{t_k}^t S_\varepsilon(t-s) (F_\varepsilon(u_\varepsilon(s)) - F(u_\varepsilon(s))) \mathbf{1}_{s < \tau_3^K} ds \right\|_1 .$$

In view of Proposition A.1 below, it thus suffices to show that, for every $p > 0$, there exists a constant C_K depending also on p such that

$$\mathbf{E}(\mathbf{1}_{s < \tau_3^K} \|F_\varepsilon(u_\varepsilon(s)) - F(u_\varepsilon(s))\|_{-1}^p) \leq C_K \varepsilon^{\frac{1-p}{2}p} . \quad (5.5)$$

In order to obtain such a bound, we write

$$F_\varepsilon(u_\varepsilon) - F(u_\varepsilon) = \varepsilon \partial_x (g(u_\varepsilon) \partial_x u_\varepsilon) + \varepsilon \bar{h}(u_\varepsilon) ((\partial_x u_\varepsilon \otimes \partial_x u_\varepsilon) - A) ,$$

where we have set $\bar{h} = h - D_s g$ (see Remark 1.2 for the meaning of $D_s g$), and A is a shortcut for $\frac{1}{2\sqrt{\nu}}$ times the identity matrix.

We then decompose u_ε as $u_\varepsilon = v_\varepsilon + \psi^\varepsilon$ and recall that $\|v_\varepsilon\|_1 \leq C_K$ by Lemma 5.7. Since, for times less than τ_3^K , the L^∞ -norm of $g(u_\varepsilon)$ and the H^1 -norm of v_ε are almost surely bounded by a constant depending on K , it follows from the bounds of Lemma 3.9 on the H^1 -norm of ψ^ε that the first term does indeed satisfy the bound (5.5). The second term can be rewritten as

$$\begin{aligned} \varepsilon \bar{h}(u_\varepsilon) ((\partial_x u_\varepsilon \otimes \partial_x u_\varepsilon) - A) &= \varepsilon \bar{h}(u_\varepsilon) ((\partial_x \psi^\varepsilon \otimes \partial_x \psi^\varepsilon) - A) \\ &\quad + 2\varepsilon \bar{h}(u_\varepsilon) (\partial_x v_\varepsilon \otimes \partial_x \psi^\varepsilon) + \varepsilon \bar{h}(u_\varepsilon) (\partial_x v_\varepsilon \otimes \partial_x v_\varepsilon) . \end{aligned}$$

The terms on the second line satisfy the bound (5.5) in the same way as before, so that it remains to consider the first term. Note that for times $t \leq \tau_3^K$, one has the almost sure bound

$$\|\bar{h}(u_\varepsilon(t))\|_\alpha \leq C_K (1 + \|\psi^\varepsilon(t)\|_\alpha) .$$

This allows us to apply Corollary 4.6, thus showing that (5.5) also holds for this term and concluding the proof. \square

As an immediate corollary of propositions 5.5, 5.6, and 5.8, we have:

Corollary 5.9 *One has $\lim_{\varepsilon \rightarrow 0} \mathbf{P}(\tau^K = \tau_1^K = \tau_2^K = \tau_3^K) = 1$ for every $K > 0$. In particular, for every choice of K , $\tau_3^K \rightarrow T \wedge \tau^*$ in probability as $\varepsilon \rightarrow 0$.*

Proof. It suffices to note that the additional *a priori* constraint enforced by any of the τ_i^K is always given as $\varrho^i(t) \leq K$, for some quantity ϱ^i for which we then obtain a bound of the type

$$\lim_{\varepsilon \rightarrow 0} \mathbf{P} \left(\sup_{t \leq \tau_i^K} \varrho^i(t) \geq \varepsilon^\beta \right) = 0 ,$$

for some exponent $\beta > 0$. Since, setting $\tau_0^K = \tau^K$, we then have $\mathbf{P}(\tau_i^K \neq \tau_{i-1}^K) = \mathbf{P}(\varrho^i(\tau_i^K) \geq K)$ for $i \in \{1, 2, 3\}$, the claim follows at once. \square

It is now straightforward to assemble all the pieces:

Proof of Theorem 5.1. Combining Propositions 5.5–5.8, we obtain

$$\lim_{\varepsilon \rightarrow 0} \mathbf{P} \left(\sup_{t \leq \tau_3^K} \|u_\varepsilon(t) - u(t)\|_{L^\infty} > \varepsilon^{\frac{1}{2} - \kappa} \right) = 0. \quad (5.6)$$

Since this is true for every $K > 0$, a standard ‘diagonal’ argument shows that there exists a sequence $K_\varepsilon \rightarrow \infty$ such that (5.6) still holds with K replaced by K_ε . Since $\tau^K \rightarrow T \wedge \tau^*$ in probability as $K \rightarrow \infty$, it follows from Corollary 5.9 that the sequence of stopping times $\tau_\varepsilon = \tau_3^{K_\varepsilon}$ has the same property as $\varepsilon \rightarrow 0$, thus proving the claim. \square

5.1 Proof of Theorem 1.5

The proof of Theorem 1.5 is slightly more straightforward than the proof of Theorem 1.1, mainly due to the fact that we gain regularity in the limit $\varepsilon \rightarrow 0$ instead of losing regularity. First of all, we reformulate the solutions as solutions to the corresponding integral equations by setting:

$$v_\varepsilon(t) = S_\varepsilon(t)v_0 + \int_0^t S_\varepsilon(t-s)G(v_\varepsilon(s))ds + \sqrt{\varepsilon}\psi^\varepsilon(t), \quad (5.7a)$$

$$v(t) = S(t)v_0 + \int_0^t S(t-s)\bar{G}(v(s))ds, \quad (5.7b)$$

where we have defined

$$G(u) = f(u) + h(u)(\partial_x u \otimes \partial_x u), \quad \bar{G}(u) = \bar{f}(u) + h(u)(\partial_x u \otimes \partial_x u),$$

with $\bar{f} = f + \frac{1}{2\sqrt{\nu}} \text{Tr} h$. This time, v_0 really is the initial condition for (5.7b), but as before it is not the initial condition for (5.7a) since $\psi^\varepsilon(0) \neq 0$. Again, we assume throughout that f and h are in \mathcal{C}^1 . It then follows from Sobolev calculus and a standard bootstrapping argument that (5.7b) is locally well-posed for initial conditions in H^1 and that its solutions, as long as they exist, belong to H^s for all $s < 3$. Therefore, for every $v_0 \in H^1$, there exists a (possibly infinite) time τ^* such that the solutions to (5.7b) exist up to time τ^* and such that, provided $\tau^* \neq \infty$, one has $\lim_{t \rightarrow \tau^*} \|v(t)\|_1 = \infty$.

With this notation, we have the following rigorous formulation of Theorem 1.5:

Theorem 5.10 *Let v_0 be an H^2 -valued random variable and fix $\beta \in (\frac{1}{2}, 1)$. Then, for every $\kappa > 0$ and every $T > 0$, there exists a sequence of stopping times τ_ε with $\tau_\varepsilon \rightarrow T \wedge \tau^*$ in probability, so that*

$$\lim_{\varepsilon \rightarrow 0} \mathbf{P} \left(\sup_{t \in [0, \tau_\varepsilon]} \|v(t) - v_\varepsilon(t)\|_{H^\beta} > \varepsilon^{1-\beta-2\kappa} \right) = 0.$$

Remark 5.11 Note that in this case, we are unable to deal with a nonlinearity of the form $g(v)\partial_x^2 v$, mainly since this would lead to a fully non-linear limiting equation, so that some of the techniques employed here break down.

Proof. Fix two exponents α, β such that $\alpha \in (1, \frac{3}{2})$ and $\beta \in (2 - \alpha, 1)$. One should think of α as being slightly less than $\frac{3}{2}$ and β as being slightly more than $\frac{1}{2}$. We also fix an exponent $\kappa < \frac{1}{4}$, we set $\varrho = v_\varepsilon - v - \sqrt{\varepsilon}\psi^\varepsilon(t)$, and we define a stopping time τ^K by

$$\tau^K = \inf\{t < T : \varepsilon^{\beta+\kappa-\frac{1}{2}}\|\psi^\varepsilon(t)\|_\beta + \sqrt{\varepsilon}\|\psi^\varepsilon(t)\|_{1-\kappa} + \|\varrho(t)\|_\alpha + \|v(t)\|_2 > K\}.$$

For $t \leq \tau^K$, one then has the bound

$$\begin{aligned} \|\varrho(t)\|_\alpha &\leq C\varepsilon^{\frac{1}{2}}\|v_0\|_2 + \left\| \int_0^t S_\varepsilon(t-s)(G(v_\varepsilon(s)) - \bar{G}(v(s))) \right\|_\alpha ds \\ &\quad + \int_0^t \|(S_\varepsilon(t-s) - S(t-s))\bar{G}(v(s))\|_\alpha ds, \\ &\leq \varepsilon^{\frac{1}{2}}(C\|v_0\|_2 + C_K) + \left\| \int_0^t S_\varepsilon(t-s)(G(v_\varepsilon(s)) - \bar{G}(v(s))) \right\|_\alpha ds, \end{aligned}$$

where we have made use of Lemma 3.6 and the fact that $\|v(s)\|_2 < K$ in order to bound the last term. Similarly, we obtain for $s < t < \tau^K$ the bound

$$\|\varrho(t)\|_\alpha \leq \|\varrho(s)\|_\alpha + C_K\varepsilon^{\frac{1}{2}} + \left\| \int_s^t S_\varepsilon(t-r)(G(v_\varepsilon(r)) - \bar{G}(v(r))) \right\|_\alpha dr.$$

Before we proceed, we obtain bounds on the last term in this inequality.

For this, we rewrite $G(v_\varepsilon) - \bar{G}(v)$ as

$$\begin{aligned} G(v_\varepsilon) - \bar{G}(v) &= (f(v_\varepsilon) - f(v)) + (h(v_\varepsilon) - h(v))(\partial_x v \otimes \partial_x v + A) \\ &\quad + h(v_\varepsilon)(\varepsilon\partial_x\psi^\varepsilon \otimes \partial_x\psi^\varepsilon - A) \\ &\quad + h(v_\varepsilon)(\partial_x\varrho \otimes (2\partial_x v + \partial_x\varrho)) \\ &\quad + 2\sqrt{\varepsilon}h(v_\varepsilon)(\partial_x\psi^\varepsilon \otimes \partial_x v) \\ &\quad + 2\sqrt{\varepsilon}h(v_\varepsilon)(\partial_x\psi^\varepsilon \otimes \partial_x\varrho) \\ &\stackrel{\text{def}}{=} I_1 + I_2 + I_3 + I_4 + I_5, \end{aligned}$$

and we bound these terms separately. Since it follows from the definition of τ^K that both v and v_ε are bounded by some constant C_K in $H^{1-\kappa}$ (and therefore in particular in L^∞), it follows that

$$\|I_1\|_{L^2} \leq C_K(\varepsilon^{1-\beta-\kappa} + \|\varrho\|_{L^2}). \quad (5.8a)$$

In order to bound I_2 , we use the fact that $\|h(v_\varepsilon)\|_\beta \leq C_K$ almost surely to apply Corollary 4.5. Therefore, for every $\gamma > \frac{1}{2}$ and every $p > 0$, we obtain constants $C_K(p)$ such that

$$\mathbf{E}\mathbf{1}_{t \leq \tau^K} \|I_2(t)\|_{-\gamma}^p \leq C_K(p)\varepsilon^{\frac{p}{2}}. \quad (5.8b)$$

In order to bound I_3 , we use the fact that both $\partial_x\varrho$ and $\partial_x v$ are bounded in L^2 , together with the fact that $L^1 \subset H^{-\gamma}$ for every $\gamma > \frac{1}{2}$, so that we have

$$\|I_3\|_{-\gamma} \leq C_K\|\varrho\|_\alpha. \quad (5.8c)$$

The bound on I_4 exploits the fact that $\sqrt{\varepsilon}\partial_x\psi^\varepsilon$ is bounded by $K\varepsilon^{1-\beta-\kappa}$ in $H^{\beta-1}$, together with the fact that $h(v_\varepsilon)\partial_x v$ is bounded by a constant in $H^{1-\kappa}$. Furthermore, the product of an element in $H^{\beta-1}$ with an element in $H^{1-\kappa}$ lies in $H^{-\frac{1}{2}}$, provided that $\kappa < \beta$ (which is always the case since we assumed that $\beta > \frac{1}{2}$), so that

$$\|I_4\|_{-\frac{1}{2}} \leq C_K \varepsilon^{1-\beta-\kappa}.$$

Finally, in order to bound I_5 , we use the fact that $\sqrt{\varepsilon}\partial_x\psi^\varepsilon$ is bounded by K in $H^{-\kappa}$, $h(v_\varepsilon)\partial_x\varrho$ is bounded by $C_K\|\varrho\|_\alpha$ in $H^{\frac{1}{2}-\kappa}$, and $\kappa < \frac{1}{4}$ to deduce that

$$\|I_5\|_{-\frac{1}{2}} \leq C_K\|\varrho\|_\alpha.$$

Making sure that the constant γ appearing in the bounds on I_2 and I_3 is less than $2 - \alpha$, we can combine all of these bounds in order to get the inequality

$$\|\varrho(t \wedge \tau^K)\|_\alpha \leq \|\varrho(s \wedge \tau^K)\|_\alpha + R_s(t) + C_K|t - s|^\delta \sup_{s \leq r \leq t} \|\varrho(r \wedge \tau^K)\|_\alpha,$$

where the remainder term R_s satisfies

$$\mathbf{E} \sup_{t \leq \tau^K} \|R_s(t)\|_\alpha \leq C_K(\varepsilon^{1-\beta-\frac{3}{2}\kappa} + \varepsilon^{\frac{1}{2}}\|v_0\|_2),$$

by Proposition A.1. We now conclude in the same way as in the proof of Proposition 5.8. \square

Appendix A Factorisation method

The aim of this section is to show the following:

Proposition A.1 *For every $\kappa > 0$ there exists $p > 0$ such that the bound*

$$\mathbf{E} \sup_{t \in [0, T]} \left\| \int_0^t S_\varepsilon(t-s) R(s) ds \right\|_\gamma^p \leq C\varepsilon^{-\kappa p} \sup_{t \in [0, T]} \mathbf{E} \|R(t)\|_{\gamma-2}^p,$$

holds for every $\gamma \in \mathbf{R}$ and every $H^{\gamma-2}$ -valued stochastic process R .

Proof. The proof relies on the ‘factorisation method’ as in [DPKZ87]. We make use of the following bounds. Recall Minkowski’s integral inequality,

$$\left(\mathbf{E} \left(\int_0^t f(s) ds \right)^p \right)^{1/p} \leq \int_0^t (\mathbf{E} f^p(s))^{1/p} ds, \quad (\text{A.1})$$

which holds for every $p \geq 1$ and every real-valued stochastic process f . Given R , we then define a process y by

$$y(t) = \int_0^t (t-s)^{-\alpha} S_\varepsilon(t-s) R(s) ds, \quad (\text{A.2})$$

where α is some (typically very small) exponent to be determined later. The convolution of R with S_ε is then given by

$$\tilde{R}(t) \stackrel{\text{def}}{=} \int_0^t S_\varepsilon(t-s) R(s) ds = C_\alpha \int_0^t (t-s)^{\alpha-1} S_\varepsilon(t-s) y(s) ds, \quad (\text{A.3})$$

where the constant C_α gets large as α gets small. It is known that for every $p > 1/\alpha$, the map $y \mapsto \tilde{R}$ given by (A.3) is bounded from $L^p([0, T], H^\gamma)$ into $\mathcal{C}([0, T], H^\gamma)$ [Hai09], so that

$$\mathbf{E} \sup_{t \in [0, T]} \|\tilde{R}(t)\|_\gamma^p \leq C \sup_{t \in [0, T]} \mathbf{E} \|y(t)\|_\gamma^p.$$

It thus suffices to show that $\mathbf{E} \|y(s)\|_\gamma^p$ is bounded by $\varepsilon^{-\kappa p} \sup_{t \in [0, T]} \mathbf{E} \|R(t)\|_{\gamma-2}^p$ for some large enough value of p .

Combining (A.2) with (A.1), we have the bound

$$(\mathbf{E} \|y(t)\|_\gamma^p)^{1/p} \leq \int_0^t (t-s)^{-\alpha} (\mathbf{E} \|S_\varepsilon(t-s)R(s)\|_\gamma^p)^{1/p} ds.$$

It then follows from Lemma 3.5 that there exists a constant C such that

$$(\mathbf{E} \|y(t)\|_\gamma^p)^{1/p} \leq C \varepsilon^{-4\alpha} \int_0^t (t-s)^{\alpha-1} (\mathbf{E} \|R(s)\|_{\gamma-2}^p)^{1/p} ds,$$

so that the required bound follows by setting $\alpha = \kappa/4$. \square

References

- [Ass02] S. ASSING. A pregenerator for Burgers equation forced by conservative noise. *Comm. Math. Phys.* **225**, no. 3, (2002), 611–632.
- [BG97] L. BERTINI and G. GIACOMIN. Stochastic Burgers and KPZ equations from particle systems. *Comm. Math. Phys.* **183**, no. 3, (1997), 571–607.
- [Bor75] C. BORELL. The Brunn-Minkowski inequality in Gauss space. *Invent. Math.* **30**, no. 2, (1975), 207–216.
- [Cha00] T. CHAN. Scaling limits of Wick ordered KPZ equation. *Comm. Math. Phys.* **209**, no. 3, (2000), 671–690.
- [DPD03] G. DA PRATO and A. DEBUSSCHE. Strong solutions to the stochastic quantization equations. *Ann. Probab.* **31**, no. 4, (2003), 1900–1916.
- [DPDT07] G. DA PRATO, A. DEBUSSCHE, and L. TUBARO. A modified Kardar-Parisi-Zhang model. *Electron. Comm. Probab.* **12**, (2007), 442–453 (electronic).
- [DPKZ87] G. DA PRATO, S. KWAPIEŃ, and J. ZABCZYK. Regularity of solutions of linear stochastic equations in Hilbert spaces. *Stochastics* **23**, no. 1, (1987), 1–23.
- [DPT07] G. DA PRATO and L. TUBARO. Wick powers in stochastic pdes: an introduction, 2007. Preprint.
- [DPZ92] G. DA PRATO and J. ZABCZYK. *Stochastic Equations in Infinite Dimensions*, vol. 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 1992.

- [Fer74] X. FERNIQUE. Des résultats nouveaux sur les processus gaussiens. *C. R. Acad. Sci. Paris Sér. A* **278**, (1974), 363–365.
- [Fre04] M. FREIDLIN. Some remarks on the Smoluchowski-Kramers approximation. *J. Statist. Phys.* **117**, no. 3-4, (2004), 617–634.
- [Hai09] M. HAIRER. An introduction to stochastic PDEs. Lecture notes, 2009. URL <http://arxiv.org/abs/0907.4178>.
- [HØUZ96] H. HOLDEN, B. ØKSENDAL, J. UBØE, and T. ZHANG. *Stochastic partial differential equations*. Probability and its Applications. Birkhäuser Boston Inc., Boston, MA, 1996. A modeling, white noise functional approach.
- [HSV07] M. HAIRER, A. M. STUART, and J. VOSS. Analysis of SPDEs arising in path sampling, part II: The nonlinear case. *Annals of Applied Probability* **17**, no. 5/6, (2007), 1657–1706.
- [HSV09] M. HAIRER, A. M. STUART, and J. VOSS. Sampling conditioned hypoelliptic diffusions, 2009. Preprint.
- [HSVW05] M. HAIRER, A. M. STUART, J. VOSS, and P. WIBERG. Analysis of SPDEs arising in path sampling, part I: The Gaussian case. *Communications in Mathematical Sciences* **3**, no. 4, (2005), 587–603.
- [JLM85] G. JONA-LASINIO and P. K. MITTER. On the stochastic quantization of field theory. *Comm. Math. Phys.* **101**, no. 3, (1985), 409–436.
- [KP92] P. E. KLOEDEN and E. PLATEN. *Numerical solution of stochastic differential equations*, vol. 23 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, 1992.
- [KPS04] R. KUPFERMAN, G. A. PAVLIOTIS, and A. M. STUART. Itô versus Stratonovich white-noise limits for systems with inertia and colored multiplicative noise. *Phys. Rev. E (3)* **70**, no. 3, (2004), 036120, 9.
- [MR10] R. MIKULEVICIUS and B. L. ROZOVSKII. On quantization of stochastic Navier-Stokes equation, 2010. Preprint.
- [RY91] D. REVUZ and M. YOR. *Continuous martingales and Brownian motion*, vol. 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1991.
- [SC74] V. N. SUDAKOV and B. S. CIREL'SON. Extremal properties of half-spaces for spherically invariant measures. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **41**, (1974), 14–24, 165. Problems in the theory of probability distributions, II.
- [Tal87] M. TALAGRAND. Regularity of Gaussian processes. *Acta Math.* **159**, no. 1-2, (1987), 99–149.
- [Tal95] M. TALAGRAND. Concentration of measure and isoperimetric inequalities in product spaces. *Inst. Hautes Études Sci. Publ. Math.* , no. 81, (1995), 73–205.
- [WZ65] E. WONG and M. ZAKAI. On the relation between ordinary and stochastic differential equations. *Internat. J. Engrg. Sci.* **3**, (1965), 213–229.