# Ergodic Properties of Markov Processes

Exercises for week 4

**Exercise 1** Find the Perron-Frobenius vectors of the matrices

$$P_1 = \frac{1}{5}\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}\;, \quad P_2 = \frac{1}{4}\begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 2 \\ 1 & 0 & 2 \end{pmatrix}\;.$$

(Normalise them so that their entries sum up to $1$.)

**Exercise 2** Is the stochastic matrix

$$P = \frac{1}{10}\begin{pmatrix} 2 & 0 & 4 & 0 \\ 0 & 3 & 0 & 5 \\ 8 & 0 & 6 & 0 \\ 0 & 7 & 0 & 5 \end{pmatrix}$$

irreducible? Aperiodic?

**Exercise 3** Show that if $P$ is irreducible and of period $p$, there is only one possible choice of $A_1, \ldots, A_p$. Show also that the period of $P$ is the smallest integer $n$ such that $P^n$ is aperiodic. What is the period of $P^n$ for general $n$?

**Exercise 4** Give an example of a stochastic matrix of period $2$ for which there are two possible choices of $A_1, A_2$. (Of course, this matrix cannot be irreducible by the previous exercise.)

**Exercise 5** Prove that if there exists $j$ such that $P_{jj} \neq 0$, then the matrix is aperiodic.

**Exercise 6** Let $P$ be an irreducible stochastic matrix. Show that $P$ is aperiodic if and only if $P^n$ is irreducible for every $n \geq 1$.

**Exercise 7** Given a vector $\mu \in \mathbf{C}^N$, we write $|\mu|$ for the vector with entries $|\mu_i|$ and $\sum(\mu)$ for the number $\sum_{i=1}^N \mu_i$. Show that if $P$ is a stochastic matrix, then one has $\sum(P\mu) = \sum(\mu)$ and $\sum(|P\mu|) \leq \sum(|\mu|)$.

*** Exercise 8** Show that the three following conditions are equivalent:

(a) $P$ is irreducible and aperiodic.

(b) $P^n$ is irreducible for every $n \geq 1$.

(c) There exists $n \geq 1$ such that $(P^n)_{ij} > 0$ for every $i, j = 1, \ldots, N$.

**Hint:** It is relatively easy to show that (c) implies (b) which implies (a). The harder part is to show that (a) implies (c). To prove this, it is helpful to consider the sets $N_i = \{k \,|\, (P^n)_{ii} > 0\}$ and to show by contradiction that (a) implies that there exists $n$ such that $\{k \geq n\} \subset N_i$ for every $i$. It is then easy to show (c) with a possibly different value of $n$.