ORNSTEIN-UHLENBECK PROCESSES ON LIE GROUPS

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Abstract. We consider Ornstein-Uhlenbeck processes (OU-processes) related to hypoelliptic diffusion on finite-dimensional Lie groups: let \( \mathcal{L} \) be a hypoelliptic, right invariant “sum of the squares” -operator on a Lie group \( G \) with associated Markov process \( X \), then we construct OU-type processes by adding horizontal gradient drifts of functions \( U \). In the natural case \( U = -\log p(1, x) \), where \( p(1, x) \) is the density of the law of the \( X_1 \) starting at the identity \( e \) with respect to the left-invariant Haar measure on \( G \), we show the Poincaré inequality by applying the Driver-Melcher inequality for “sum of the squares” operators on Lie groups.

The Markov process associated to \( -\log p(1, x) \) is called the OU-process related to the given hypoelliptic diffusion on \( G \). We prove the global strong existence of this OU-process on \( G \). The Poincaré inequality for a large class of potentials \( U \) is then shown by perturbation methods and used to obtain a hypoelliptic equivalent of the standard result on cooling schedules for simulated annealing. The relation between local results on \( \mathcal{L} \) and global results for the constructed OU-process is widely used in this study.

1. Preparations from functional analysis

We consider a finite-dimensional Lie group \( G \) with Lie algebra \( \mathfrak{g} \), its right-invariant Haar measure \( \mu \) and a family of left-invariant vector fields \( V_1, \ldots, V_d \in \mathfrak{g} \). We assume that Hörmander’s condition holds, i.e. the sub-algebra generated by \( V_1, \ldots, V_d \) coincides with \( \mathfrak{g} \).

We consider furthermore a stochastic basis \((\Omega, \mathcal{F}, P)\) with a \( d \)-dimensional Brownian motion \( B \) and the Lie group valued process

\[
\frac{dX^x_t}{dt} = \sum_{i=1}^{d} V_i(X^x_t) \circ dB^i_t, \quad X^x_0 = x \in G.
\]

The generator of this process is denoted by \( \mathcal{L} \) and we have

\[
\mathcal{L} = \frac{1}{2} \sum_{i=1}^{d} V_i^2,
\]

where we interpret the vector fields as first order differential operators on \( C^\infty(G, \mathbb{R}) \). Furthermore, we define a semigroup \( P_t \) acting on bounded measurable functions \( f : G \to \mathbb{R} \) by

\[
P_t f(x) = E(f(X^x_t)) .
\]

This semigroup can be extended to a strongly continuous semigroup on \( L^2(G, \mu) \), which we will denote by the same letter \( P_t \). The carré du champ operator \( \Gamma \) is defined for functions \( f \), where it makes sense, by

\[
\Gamma(f, g) = \mathcal{L}(fg) - f\mathcal{L}g - g\mathcal{L}f .
\]
In our particular case, we obtain immediately
\[ \Gamma(f, f) = \sum_{i=1}^{d} (V_i f)^2. \]

Notice that the carré du champ operator does not change if we add a drift to the generator \( \mathcal{L} \).

Due to the right invariance of the Haar measure \( \mu \) and the left-invariance of the vector fields \( V_i \), the operator \( \mathcal{L} \) is symmetric (reversible) with respect to \( \mu \) and therefore \( \mu \) is an infinitesimal invariant measure in the sense that \( \int \mathcal{L} f(x) \mu(dx) = 0 \) for all smooth compactly supported functions \( f \). Furthermore, due to the symmetry of \( \mathcal{L} \) and the invariance of \( \mu \), we have from (1.2) the relation
\[ 2 \int f \mathcal{L} g \mu = - \int \Gamma(f, g) \mu \quad (1.3) \]
for all \( f, g \in C_0^\infty(G) \).

Let now \( U : G \to \mathbb{R} \) be an arbitrary smooth function such that
\[ \int_G \exp(-U(x)) \mu(dx) < \infty, \]
and consider the modified generator
\[ \mathcal{L}^U := \mathcal{L} - \frac{1}{2} \Gamma(U, \cdot) \).

Notice that \( \mu^U = \exp(-U) \mu \) is an invariant (finite) measure for \( \mathcal{L}^U \), since, by (1.3),
\[ \int \mathcal{L}^U f \mu^U = \int (\mathcal{L} f) \exp(-U) \mu - \frac{1}{2} \int \Gamma(U, f) \exp(-U) \mu \]
\[ = -\frac{1}{2} \int \Gamma(f, \exp(-U)) \mu - \frac{1}{2} \int \Gamma(f, U) \exp(-U) \mu = 0. \]

Here, the last step uses the fact that \( \Gamma(f, \cdot) \) is a derivation.

We have the following observation on the existence of a spectral gap at 0:

**Proposition 1.1.** The operator \( \mathcal{L}^U \) has a spectral gap at 0 of size \( a > 0 \) if and only if
\[ \int_G \Gamma(f, f) \mu^U(dx) \geq 2a \int_G f(x)^2 \mu^U(dx), \]
for all compactly supported smooth functions \( f \) on \( G \) satisfying
\[ \int_G f(x) \mu^U(dx) = 0. \]

**Proof.** \( \mathcal{L}^U \) has a spectral gap of size \( a > 0 \) at 0 if
\[ \int_G f(x) \mathcal{L}^U f(x) \mu^U(dx) \leq -a \int_G f(x)^2 \mu^U(dx) \]
for test functions with \( \int_G f(x) \mu^U(dx) = 0 \). Now, by construction, the integral on the left hand side can be partially integrated, hence
\[ -\frac{1}{2} \int_G \Gamma(f, f) \mu^U(dx) \leq -a \int_G f(x)^2 \mu^U(dx) \]
for any test function satisfying the integral constraint. This proves the desired inequality. \( \square \)
Remark 1.2. Assume that there is a spectral gap. Then the largest $a > 0$ in the previous inequality is the modulus of the smallest non-zero spectral value of $L_U$.

Remark 1.3. If we want to write an inequality for all test functions $f$, it reads like

$$\int_G \Gamma(f, f)(x) \mu_U(dx) \geq 2a \left( \int_G f(x)^2 \mu_U(dx) \int_G \mu_U(dx) - \left( \int_G f(x) \mu_U(dx) \right)^2 \right)$$  \hspace{1cm} (1.4)

for all test functions $f \in C^\infty_0(G)$.

2. Strong existence of OU-processes with values in Lie groups

We consider now the special case where we take as our ‘potential’ $W_t(x) = -\log p_t(t, x)$, $t > 0$, where $p_t(t, x)$ is the density of the law of $X_t$ with respect to $\mu$. By Hörmander’s Theorem [Hör67, Hör07], the function $(t, x) \mapsto p(t, x)$ is a positive and smooth function, hence the potential $W_t$ is as in the previous section. We write for short $L_t$ instead of $L_{W_t}$ and we call the associated Markov process the OU-process on $G$. We show that we have in fact global strong solutions for the corresponding Stratonovich SDE with values in $G$. The next proposition is slightly more general.

Proposition 2.1. Consider a smooth potential $U : G \to \mathbb{R}$ such that

$$\int \exp(-U(x)) \mu(dx) < \infty.$$  

Consider the following Stratonovich SDE with values in $G$:

$$dY^y_t = V_0(Y^y_t)dt + \sum_{i=1}^d V_i(Y^y_t) \circ dB^i_t, \quad Y^y_0 = y \in G, \quad \hspace{1cm} (2.1)$$

where $V_0 f = -\frac{1}{2} \Gamma(U, f)$ for smooth test functions $f$. Then there is a global strong solution to (2.1) for all initial values $y \in G$.

Proof. Since the coefficients defining (2.1) are smooth by assumption, there exists a unique strong solution up to the explosion time

$$\zeta_y = \inf\{t : \lim_{\tau \to t} Y^y_\tau = \infty\}.$$  

We then define a semigroup $P_t$ on $L^2(G, \mu_U)$ by

$$(P_t f)(y) = E(f(Y^y_t) 1_{\zeta_y > t}) \quad \hspace{1cm} (2.2)$$

It can be shown in the exact same way as in [Che73, Li92] that $P_t$ is a strongly continuous contraction semigroup and that its generator $A$ coincides with $L_U$ on the set $C^\infty_0(G)$ of compactly supported smooth functions.

On the other hand, setting $D(L_U) = C^\infty_0(G)$, one can show as in [Che73, Li92] that $L_U$ is essentially self-adjoint, so that one must have $A = L^*_U = (L^*_U)^*$. In particular, since the constant function $1$ belongs to $L^2(G, \mu_U)$ by the integrability of $\exp(-U)$ and since $\int (L^*_U \psi)(x) \mu_U(dx) = 0$ for any test function $\psi \in C^\infty_0(G)$, $1$ belongs to the domain of $(L^*_U)^*$ and therefore also to the domain of $A$. This then implies that $\mathbb{P}_t 1 = 1$ by the same argument as in [Li92]. In particular, coming back to the definition (2.2) of $P_t$, we see that $\mathbb{P}(\zeta_y = \infty) = 1$ for every $y$, which is precisely the non-explosion result that we were looking for. \hfill \square
Remark 2.2. While this argument shows that, given a fixed initial condition $y$, there exists a unique global strong solution $Y^y_t$ to (2.1), it does not prevent more subtle kinds of explosions, see for example [Elw78].

By Proposition 2.1 and since $p(t, x)$ is smooth and integrable, it follows immediately that the OU-process exists globally in a strong sense.

Corollary 2.3. For any given $\tau > 0$, the process

$$dY^y_t = V_0(Y^y_t)dt + \sum_{i=1}^d V_i(Y^y_t) \circ dB^i_t, \quad Y^y_0 = y \in G,$$

with $V_0f = -\frac{1}{2} \Gamma(W, f)$ has a global strong solution.

Remark 2.4. More traditional Lyapunov-function based techniques seem to be highly non-trivial to apply to this situation, due to the lack of information on the behaviour of $U(y)$ at large $y$. In view of [BA88, Léa87b, Léa87a], it is tempting to conjecture that one has the asymptotic

$$\lim_{t \to 0} t^2 \log \partial_t p(t, y) = -d^2(e, y), \quad (2.3)$$

and that the limit holds uniformly over compact sets $K$ that do not contain the origin $e$. (Note that it follows from [BA88] that this is true provided that $K$ does not intersect the cut-locus.) If it were the case that (2.3) holds, the space-time scaling properties of $p(t, x)$ would imply that, for every $t > 0$, there exists a compact set $K$ such that $\mathcal{L}p(t, x) = \partial_t p(t, x) > 0$ for $x \notin K$. On the other hand, one has

$$\mathcal{L}^t W_t = -\frac{1}{2} \Gamma(W_t, W_t) + \mathcal{L}W_t = -\frac{\mathcal{L}p(t, \cdot)}{p(t, \cdot)},$$

so that this would imply that $W_t$ is a Lyapunov function for the corresponding OU-process.

3. Spectral Gaps for OU-processes

Next we consider the question if $\mathcal{L}^t$ admits a spectral gap, which turns out to be a consequence of the Driver-Melcher inequality.

Theorem 3.1. The following assertions are equivalent:

- The operator $\mathcal{L}^t$ has a spectral gap of size $a_t \geq 0$ for $t > 0$ and a positive $H^1$-function $a : \mathbb{R}_+ \to \mathbb{R}_+$.
- The local estimate

$$P_t(\Gamma(f, f))(g) \geq 2a_t((P_t f^2)(g) - ((P_t f)(g))^2)$$

holds true for all test functions $f : G \to \mathbb{R}$, $t > 0$ and a positive $H^1$-function $a : \mathbb{R}_+ \to \mathbb{R}_+$ at one (and therefore all) point $g \in G$.

Furthermore, if we know that

$$\Gamma(P_t f, P_t f)(e) \leq \varphi(t)P_t(\Gamma(f, f))(e)$$

holds true for all test functions $f \in C_0^\infty(G)$, all $t \geq 0$, and a strictly positive locally integrable function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, then we can choose $a_t$ by

$$a_t \int_0^t \varphi(t-s)ds = \frac{1}{2},$$

for $t > 0$ and the two equivalent assertions hold true.
Proof. Since \( \mu^{W_t} \) is equal to the law of \( X^t_e \), one has \( \int f \mu^{W_t} = P_t f(e) \) for every \( f \in C_0^\infty(G) \). The equivalence of the first two statements then follows from (1.4) and the fact that the translation invariance of (1.1) implies that if the bound holds at some \( g \), it must hold for all \( g \in G \). We fix a test function \( f : G \to \mathbb{R} \) as well as \( t > 0 \) and consider
\[
H(s) = P_s((P_{t-s}f)^2)
\]
for \( 0 \leq s \leq t \). The derivative of this function equals
\[
H'(s) = P_s(\Gamma(P_{t-s}f, P_{t-s}f))
\]
and therefore – assuming the third statement – we obtain
\[
H'(s) \leq \varphi(t-s)P_t(\Gamma(f, f)).
\]
Whence we can conclude
\[
H(t) - H(0) \leq \int_0^t \varphi(t-s)dsP_t(\Gamma(f, f)),
\]
which is the second of the two equivalent assertions for an appropriately chosen \( a \). \( \square \)

Remark 3.2. We can replace the Lie group \( G \) by a general manifold \( M \), on which we are given a hypo-elliptic, reversible diffusion \( X \) with “sum of the squares” generator \( L \). Then the completely analogous statement holds on \( M \), in particular local Poincaré inequalities on \( M \) for \( L \) lead to a spectral gap for the OU-type process \( L^t \) with \( t > 0 \).

Corollary 3.3. Let \( G \) be a free, nilpotent Lie group with \( d \) generators \( e_1, \ldots, e_d \) and denote by \( X \) the canonical diffusion process on \( G \), i.e.
\[
dX_t = \sum_{i=1}^d X_t e_i \circ dB_t^i.
\]
Then the operator \( L^t \) has a spectral gap of size \( a_t = \frac{1}{2Kt} \) for some constant \( K > 0 \).

Proof. Due to the very interesting thesis [DM07], there is a constant \( K \) such that the bound
\[
\Gamma(P_t f, P_t f)(e) \leq KP_t(\Gamma(f, f))(e)
\]
holds true for all test functions \( f \in C_0^\infty(G) \) and for all times \( t \geq 0 \). Furthermore, due to the scaling properties of \( P_t \), there exists a best constant \( K \) such that this bound is sharp for all \( t \geq 0 \). This shows that \( a_t Kt = \frac{1}{2} \), due to the assertions of Theorem 3.1. \( \square \)

3.1. Generalisation to homogeneous spaces. Consider now \( M \) a homogeneous space with respect to the Lie group \( G \), i.e. we have a (right) transitive action \( \hat{\pi} : G \times M \to M \) of \( G \) on \( M \). We assume that there exists a measure \( \mu^M \) on \( M \) which is invariant with respect to this action. We also assume that we are given a family \( V_1, \ldots, V_d \) of left-invariant vector fields on \( G \) that generate its entire Lie algebra \( g \) as before. These vector fields induce vector fields \( V_i^M \) on \( M \) by means of the action \( \hat{\pi} \).
By choosing an ‘origin’ \( o \in M \), we obtain a surjection \( \pi: G \to M \) by \( \pi(g) = \hat{\pi}(g, o) \). Due to the invariance of \( \mu^M \) with respect to the action \( \hat{\pi} \), the vector fields \( V_i^M \) are anti-symmetric operators on \( L^2(\mu^M) \) and the generator

\[
\mathcal{L}^M = \frac{1}{2} \sum_{i=1}^{d} (V_i^M)^2
\]

is consequently symmetric on \( L^2(\mu^M) \). In particular we have

\[
(V_i^M f) \circ \pi = V_i(f \circ \pi)
\]

for \( i = 1, \ldots, d \). The local Driver-Melcher inequality on \( G \) translates to the same inequality on \( M \) by means of

\[
P_t^M(f) \circ \pi = P_t(f \circ \pi)
\]

for test functions \( f: M \to \mathbb{R} \), hence we obtain the corresponding Driver-Melcher inequality on \( M \).

4. A SIMPLE RESULT ON SIMULATED ANNEALING

For applications to simulated annealing, our main tool will be the following simple Theorem:

**Theorem 4.1.** Let \( U: G \to \mathbb{R} \) be a potential \( U \) with

\[
|U + \log p(\varepsilon, \cdot)| \leq D_\varepsilon
\]

for some \( \varepsilon > 0 \) and some constant \( D_\varepsilon > 0 \). Assume furthermore that a Poincaré inequality holds for \( \mathcal{L}^\varepsilon \), i.e.

\[
P_\varepsilon(f^2)(e) \leq K_\varepsilon P_\varepsilon(\Gamma(f,f))(e) \quad (4.1)
\]

for test functions \( f \in C_0^\infty(G) \) with \( P_\varepsilon f(e) = 0 \) and some constant \( K > 0 \). Then one has \( \exp(-U) \in L^1(\mu(dx)) \) and the Poincaré inequality

\[
\int f^2(x) \exp(-U(x)) \mu(dx) \leq C_\varepsilon \int \Gamma(f,f)(x) \exp(-U(x)) \mu(dx) \quad (4.2)
\]

holds for all test functions \( f \in C_0^\infty(G) \) with \( \int f(x) \exp(-U(x)) \mu(dx) = 0 \) and some constant \( C_\varepsilon = K_\varepsilon \exp(2D_\varepsilon) > 0 \). In particular, this leads to a spectral gap of size at least \( \frac{1}{C_\varepsilon^2} \) for \( \mathcal{L}^U \).

**Proof.** It follows immediately from the inequality

\[
p(\varepsilon, x) = \exp(-U(x)) \exp(U(x)) p(\varepsilon, x) \geq \exp(-D) \exp(-U(x))
\]

that \( \exp(-U) \in L^1(\mu) \). Furthermore,

\[
\exp(-U(x)) = \frac{1}{p(\varepsilon, x) \exp(U(x))} p(\varepsilon, x) \geq \exp(-D) p(\varepsilon, x)
\]

for all \( x \in G \) by assumption. Hence we deduce (4.2) with \( C_\varepsilon = K_\varepsilon \exp(2D_\varepsilon) \) from (4.1).

**Remark 4.2.** See [BLW07] for results on unbounded perturbations, where one can hope for similar conclusions.
Throughout the remainder of this section we assume that $M$ is a nilmanifold, that is a homogeneous space with respect to a nilpotent Lie group $G$. We consider the same structures as in Section 3.1 on $M$, but we omit the index $M$ in the vector fields and measures in order to improve readability. We shall furthermore assume that $M$ satisfies the following global estimate:

**Assumption 4.3.** There is a constant $\tilde{D}$ such that

\[ |d(x_0, x)^2 + t \log p(t, x_0, x)| \leq \tilde{D} \]  

for all $0 < t < 1$ and all $x \in M$ and some (and therefore all by translation invariance) $x_0 \in M$. Here, $d$ denotes the lift of the Carnot-Carathéodory distance from $G$ to $M$.

**Remark 4.4.** If $M$ is a compact nilmanifold, we can apply Léandre’s beautiful result [Léa87b, Léa87a] that

\[ \lim_{t \to 0} t \log p(t, x_0, x) = -d(x_0, x)^2 \]

holds true uniformly on $M$, which implies Assumption 4.3.

A non-compact example of a nilmanifold, where this estimate still holds true is given by the Heisenberg group $G^2$. Notice that this is an example of dimension $d + \frac{d(d-1)}{2}$.

We prove a quantitative simulated annealing result under the previous assumption on the nilmanifold $M$. The idea is to introduce a parameter $\varepsilon$ in the operators,

\[ \mathcal{L}^{U, \varepsilon} = \mathcal{L} - \frac{1}{2} \Gamma(\frac{U}{\varepsilon^2}, \cdot), \]

such that the associated invariant measure $\exp(-\frac{U}{\varepsilon^2})\mu$ concentrates around the minima of $U$. Notice that the previous operator satisfies

\[ \varepsilon^2 \mathcal{L}^{U, \varepsilon} = \varepsilon^2 \mathcal{L} - \frac{1}{2} \Gamma(U, \cdot), \]

hence a spectral gap for $\varepsilon^2 \mathcal{L}^{U, \varepsilon}$ leads to a spectral gap for the diffusion process

\[ dY^y_t = V_0(Y^y_t)dt + \sum_{i=1}^{d} \varepsilon V_i(Y^y_t) \circ dB^i_t, \quad Y^y_0 = y \in G, \]

with $V_0f = -\frac{1}{2} \Gamma(U, f)$. Consequently we know – given strong existence – that the law of $Y^y_t$ converges to $\exp(-\frac{U}{\varepsilon^2})\mu$ and concentrates a posteriori around the minima of $U$.

In the following theorem we try to quantify this behaviour. We denote by $\mu^{U, \varepsilon}$ the probability measure invariant for $\mathcal{L}^{U, \varepsilon}$ and we use the notation

\[ \var\varepsilon(f) = \langle (f - \langle f \rangle)^2 \rangle \varepsilon \]

with $\langle f \rangle \varepsilon = \int_M f(g) \mu^{U, \varepsilon}(dg)$.

**Theorem 4.5.** Let $U : M \to \mathbb{R}$ be a smooth function such that there exist a constant $D$ and a point $x_0 \in M$ such that

\[ |U(x) - d(x_0, x)^2| \leq D, \]

for all $x \in M$. Then there exist constants $R, c > 0$ such that for $\varepsilon(t) = \frac{\varepsilon}{\sqrt{\log(R+t)}}$ and

\[ \var\varepsilon(t)(f) \leq K(R + t)\langle \Gamma(f, f) \rangle \varepsilon(t), \]
for all test functions $f \in C_0^\infty(M)$ and $t \geq 0$.

Proof. We can start to collect results. Combining Corollary 3.3 and Assumption 4.3 with Theorem 4.1, we obtain that spectral gap for the operator $L^U,\varepsilon$ has size at least

$$\frac{1}{K \varepsilon^2} \exp\left(-\frac{2(D + \tilde{D})}{\varepsilon^2}\right)$$

for $0 < \varepsilon < 1$, so that $\varepsilon^2 L^U,\varepsilon$ has a spectral gap of size at least

$$\frac{1}{K} \exp\left(-\frac{2(D + \tilde{D})}{\varepsilon^2}\right).$$

We choose $\varepsilon^2 = 2(D + \tilde{D})$ and $R$ sufficiently big so that $\varepsilon(t) < 1$ for $t \geq 0$, and we conclude that

$$K \exp\left(\frac{2(D + \tilde{D})}{\varepsilon(t)^2}\right) \leq K(R + t)$$

for all $t \geq 0$, which is the desired result.

We denote by $Z$ the process with cooling schedule $t \mapsto \varepsilon(t)$ as in the previous theorem,

$$dZ^z_t = V_0(Z^z_t)dt + \sum_{i=1}^d \varepsilon(t) V_i(Z^z_t) \circ dB^i_t.$$

Then the previous conclusion leads to the following Proposition.

**Proposition 4.6.** Let $f_t$ denote the Radon-Nikodym derivative of the law of $Z^z_t$ with respect to $\mu_t = \mu^U,\varepsilon(t)$ and let

$$u(t) := \|f_t - 1\|^2_{L^2(\mu_t)}$$

denote the distance in $L^2(\mu_t)$, then

$$u'(t) \leq -\frac{1}{K(R + t)} u(t) + \frac{1}{c^2(R + t)} u(t) + \frac{1}{c^2(R + t)} \sqrt{u(t)}$$

for the constants $R, c$ and $K$ from Theorem 4.5.

**Remark 4.7.** If we choose $c^2 > K$, which is always possible, we conclude that $\sup_{t \geq 0} u(t)$ is bounded from above by a constant depending on $f_0, c, R$ and $K$.

Proof. The proof follows closely the lines of [HS88]. By assumption we know that $u(t) = ||f_t||^2_{L^2(\mu_t)} - 1$ and hence with the notation $\beta(t) = \frac{1}{\varepsilon(t)^2}$,

$$u'(t) = -2\langle\Gamma(f_t, f_t)\rangle_{\varepsilon(t)} - \beta'(t) \int (U - \langle U \rangle_{\varepsilon(t)}) f_t^2 \mu_t$$

$$= -2\langle\Gamma(f_t, f_t)\rangle_{\varepsilon(t)} - \beta'(t) \int (U - \langle U \rangle_{\varepsilon(t)})(f_t - 1)^2 \mu_t$$

$$- 2\beta'(t) \int (U - \langle U \rangle_{\varepsilon(t)})(f_t - 1) \mu_t$$

$$\leq -\frac{1}{K(R + t)} u(t) + \frac{1}{c^2(R + t)} u(t) + \frac{1}{c^2(R + t)} \sqrt{u(t)}.$$

Here, we used the Cauchy-Schwarz inequality and the conclusions of the previous Theorem 4.5. □
Theorem 4.8. Assume that we are in the previous settings with $c^2 > K$, so that 
$\sup_{t \geq 0} \| f_t \|_{L^2(\mu_t)} \leq M < \infty$. Define $U_0 = \inf_{x \in M} U(x)$ and, for every $\delta > 0$, denote by $A_\delta$ the set $A_\delta = \{ x \in M \mid U(x) \geq U_0 + \delta \}$. Then we can conclude that 
$P(Z_t \in A_\delta) \leq M \sqrt{\mu_t(A_\delta)}$
for every $t > 0$ and every $\delta \geq 0$.

Proof. It follows from the Cauchy-Schwarz inequality that 
$P(Z_t \in A_\delta) = \int_{A_\delta} f_t \mu_t \leq M \sqrt{\mu_t(A_\delta)}$, as required. \qed

Remark 4.9. Since $\lim_{\varepsilon \to 0} \mu_{U,\varepsilon}(A_\delta) = 1$ for every $\delta > 0$, we obtain that for all continuous bounded test functions $f$, we have 
$E(f(Z_t)) \to f(x_0)$, 
provided that there is only one element $x_0 \in M$ such that $U(x_0) = U_0$.

References


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