Regularity Structures

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What are regularity structures?

Algebraic structures providing "skeleton" for analytical "models" mimicking properties of Taylor polynomials: (T, G, A).
Model: $T \times \mathbf{R}^d \to \mathcal{D}'$.

Polynomial model: $(P, x_0) \mapsto P(\cdot - x_0)$.

Algebraic properties: Group G acting by reexpansions on $P \in T$:

$$P(x - x_0) = P((x - x_1) + x_1 - x_0) = \big(\Gamma_{x_0,x_1} P\big)(x - x_1) \ .$$

For every $\Gamma \in G$, $\deg(\Gamma P - P) < \deg P$ and $\Gamma PQ = (\Gamma P)(\Gamma Q)$.

Analytical properties: Homogeneous monomials vanish at base point with order (speed) equal to their degree.

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Another example

T: linear span of $\mathbf{1}$ (degree 0) and $\mathbf{W}$ (degree $\frac{1}{2}$).

Model: For some fixed Hölder-$\frac{1}{2}$ function W, set

$$(a\mathbf{1} + b\mathbf{W}, x_0) \mapsto a + b\big(W(\cdot) - W(x_0)\big) \;.$$

Group G: $\Gamma_{x_0,x_1}\mathbf{W} = \mathbf{W} + \big(W(x_0) - W(x_1)\big)\mathbf{1}$.
$\Gamma_{x_0,x_1}\mathbf{1} = \mathbf{1}$

Important property: For a given regularity structure, one can have many different models. (Here: given by choice of W.)

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What are they good for?

Construct robust solution theories for very singular SPDEs.
Examples:

$$\begin{aligned}
\partial_t h &= \partial_x^2 h + (\partial_x h)^2 + \xi\;, && (d=1)\\
\partial_t \Phi &= \Delta\Phi - \Phi^3 + \xi\;, && (d=2,3)\\
\partial_t u &= \Delta u + g_{ij}(u)\partial_i u\,\partial_j u + \sigma(u)\eta\;, && (d=2,3)\\
\partial_t v &= \partial_x^2 v + f(v) + \sigma(v)\xi\;. && (d=1)
\end{aligned}$$

Here ξ is space-time white noise and η is spatial white noise.

KPZ (h): universal model for interface propagation. Dynamical Φ^4_3: universal model for dynamics of near mean-field phase transition models near critical temperature. PAM (u with $g=0$ and $\sigma(u)=u$): universal model for weakly killed diffusions.

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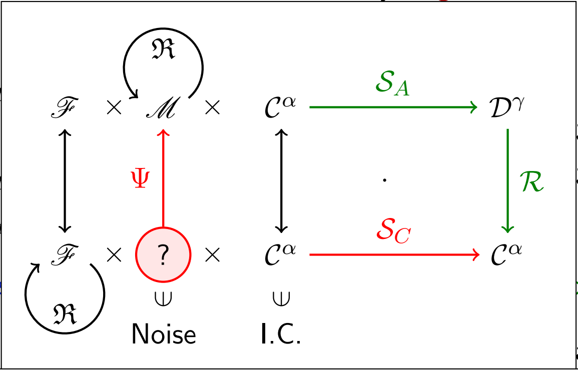

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Φ^4_3: uni…

transiti… $= 0$

Cf. "A Theory of Regularity Structures" (H. '14)

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Example of renormalisation

Try to define distribution "$\eta(x) = \frac{1}{|x|} - C\delta(x)$".

Problem: Integral of $1/|x|$ diverges, so we need to set "$C = \infty$" to compensate!

Formal definition:

$$\eta_\chi(\phi) = \int_{\mathbf{R}} \frac{\phi(x) - \chi(x)\phi(0)}{|x|}\,dx\;,$$

for some smooth compactly supported cut-off χ with $\chi(0) = 1$. Yields one-parameter family $c \mapsto \eta_c$ of models, but no canonical "choice of origin" for c.

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Previous notions of solution

1. If nonlinear term is $\sigma(u)\,\xi$, Itô calculus can be used. Relies crucially on martingale property, broken by regularisation.
2. KPZ and $1D$ stochastic Burgers can be treated using controlled rough paths by Lyons / Gubinelli (H. '11 / H. '13).
3. Solve $\partial_t Z = \partial_x^2 Z + Z\,\xi$ (SHE) and interpret $h = \log Z$ as KPZ (Hopf '50 / Cole '51 / Bertini-Giacomin '97).
4. Dynamical Φ^4_2 model: write $\Phi = \Psi + \tilde\Phi$ with Ψ solution to linear equation and derive well-posed equation for $\tilde\Phi$ (Albeverio-Röckner '91 / Da Prato-Debussche '03).
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Universality

Central limit theorem: Gaussian universality

KPZ strong Universality conjecture: At large scales, the space-time fluctuations of a large class of $1+1$-dimensional interface propagation model are described by a universal Markov process H, self-similar with exponents $1-2-3$:

$$\lambda^{-1} H(\lambda^2 x, \lambda^3 t) \stackrel{\text{law}}{=} H(x,t) .$$

Exactly solvable models: Borodin, Corwin, Quastel, Sasamoto, Spohn, etc. Yields partial characterisation of limiting "KPZ fixed point" (H): agrees with experimental evidence (Takeuchi & Al).

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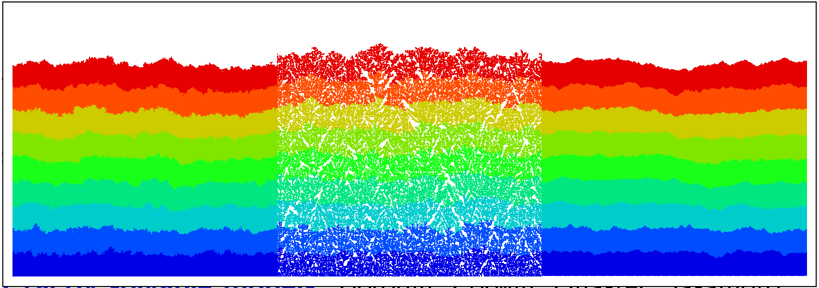

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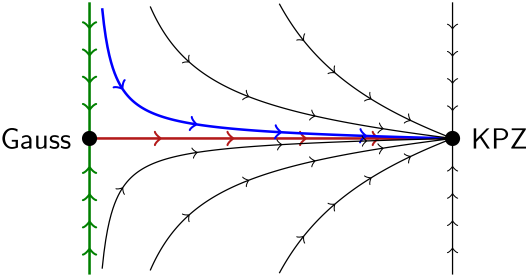
Heuristic picture

Universality for symmetric interface fluctuation models: scaling exponents $1-2-4$, Gaussian limit. Heuristic picture of the evolution of interface models under "zooming out":



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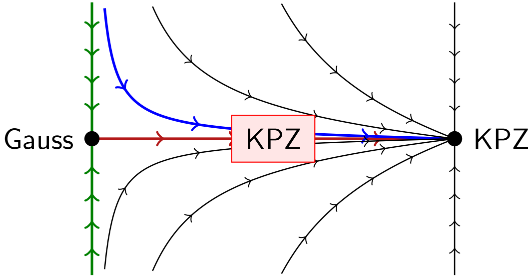

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Weak Universality conjecture

Conjecture: the KPZ equation is the only model on the "red line".

Conjecture: Let h_ε be any "natural" one-parameter family of asymmetric interface models with ε denoting the strength of the asymmetry such that propagation speed $\approx \sqrt{\varepsilon}$.

As $\varepsilon \to 0$, there is a choice of $C_\varepsilon \sim \varepsilon^{-1}$ such that $\sqrt{\varepsilon} h_\varepsilon(\varepsilon^{-1}x, \varepsilon^{-2}t) - C_\varepsilon t$ converges to solutions h to the KPZ equation.

Height function of WASEP (Bertini-Giacomin '97).
Accumulation points satisfy weak version of KPZ for certain generalisations of WASEP (Jara-Gonçalves '10).

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Weak universality result for KPZ

Class of models:

$$\partial_t h_\varepsilon = \partial_x^2 h_\varepsilon + \sqrt{\varepsilon} P(\partial_x h_\varepsilon) + F \;,$$

with P an even polynomial, F a Gaussian field with compactly supported correlations $\rho(t,x)$ s.t. $\int \rho = 1$.

Theorem (H., Quastel '14, in progress) As $\varepsilon \to 0$, there is a choice of $C_\varepsilon \sim \varepsilon^{-1}$ such that $\sqrt{\varepsilon} h(\varepsilon^{-1}x, \varepsilon^{-2}t) - C_\varepsilon t$ converges to solutions to $(\mathrm{KPZ})_\lambda$ with λ depending in a non-trivial way on all coefficients of P.

Remark: Convergence to KPZ with $\lambda \neq 0$ even if $P(u) = u^4$!!

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Nonlinearity $\lambda(\partial_x h)^2$

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Case $P(u) = u^4$

Write $\tilde{h}_\varepsilon(x,t) = \sqrt{\varepsilon} h(\varepsilon^{-1}x, \varepsilon^{-2}t) - C_\varepsilon t$. Satisfies

$$\partial_t \tilde{h}_\varepsilon = \partial_x^2 h_\varepsilon + \varepsilon(\partial_x \tilde{h}_\varepsilon)^4 + \xi_\varepsilon - C_\varepsilon \;,$$

with ξ_ε an ε-approximation to white noise.

Fact: Derivatives of microscopic model do not converge to 0 as $\varepsilon \to 0$: no small gradients! Heuristic: gradients have $\mathcal{O}(1)$ fluctuations but are small on average over large scales... General formula:

$$\lambda = \frac{1}{2} \int P''(u)\, \mu(du) \;, \qquad C_\varepsilon = \frac{1}{\varepsilon} \int P(u)\, \mu(du) + \mathcal{O}(1) \;,$$

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Main step in proof

Rewrite general equation in integral form as

$$H = \mathcal{P}\big(\mathcal{E}(\mathscr{D}H)^4 + a(\mathscr{D}H)^2 + \Xi\big)\;,$$

with $\mathcal{E}$ an abstract "integration operator" of order 1, $\mathcal{P}$ convolution with heat kernel.

Find two-parameter lift of noise $\xi_\varepsilon \mapsto \Psi_{\alpha,c}(\xi_\varepsilon)$ so that $h = \mathcal{R}H$ solves

$$\begin{aligned}\partial_t h &= \partial_x^2 h + \alpha H_4(\partial_x h, c) + aH_2(\partial_x h, c) + \xi_\varepsilon \\ &= \partial_x^2 h + \alpha(\partial_x h)^4 + (a - 6\alpha c)(\partial_x h)^2 + (3\alpha c^2 - ac) + \xi_\varepsilon\;.\end{aligned}$$

Show that $\Psi_{\varepsilon,1/\varepsilon}(\xi_\varepsilon)$ converges to same limit as $\Psi_{0,1/\varepsilon}(\xi_\varepsilon)$! (Actually more complicated: logarithmic sub-divergencies...)

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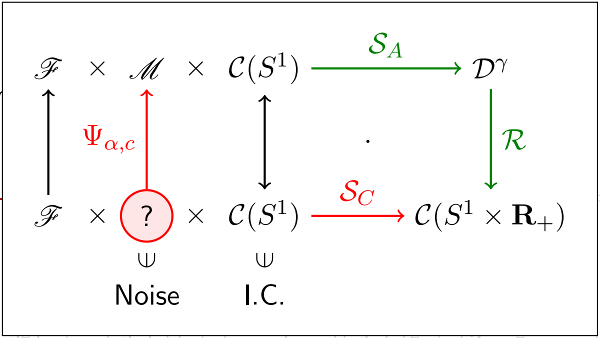

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Show that $\Psi_{\varepsilon,1/\varepsilon}(\xi_\varepsilon)$ converges to same limit as $\Psi_{0,1/\varepsilon}(\xi_\varepsilon)$!
(Actually more complicated: logarithmic sub-divergencies...)

Main step in proof

Rewrite general equation in integral form as

$$H = \mathcal{P}\big(\mathcal{E}(\mathscr{D}H)^4 + a(\mathscr{D}H)^2 + \Xi\big)\;,$$

with $\mathcal{E}$ an abstract "integration operator" of order 1, $\mathcal{P}$ convolution with heat kernel.

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Many open questions remain:

1. Strong Universality without exact solvability???
2. Hyperbolic / dispersive problems??
3. Obtain convergence results for discrete models (H.-Maas-Weber '12; Mourrat-Weber, in progress).
4. Non-Gaussian noise / fully nonlinear continuum models.
5. Control over larger scales $\Rightarrow$ KPZ fixed point.
6. Characterisation of possible renormalisation maps. When does it yield a modified equation in closed form?
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