comes to the definition of a rough integral we typically take $\bar{W} = \mathcal{L}(V, W)$; although other choices can be useful (see e.g. remark 4.11). In the context of rough differential equations, with solutions in $\bar{W} = W$, we actually need to integrate $f(Y)$, which will be seen to be controlled by X for sufficiently smooth coefficients $f : W \to \mathcal{L}(V, W)$.

Definition 4.6. Given a path $X \in \mathcal{C}^\alpha([0, T], V)$, we say that $Y \in \mathcal{C}^\alpha([0, T], \bar{W})$ is *controlled* by X if there exists $Y' \in \mathcal{C}^\alpha([0, T], \mathcal{L}(V, \bar{W}))$ so that the remainder term R^Y given implicitly through the relation

$$Y_{s,t} = Y'_s X_{s,t} + R^Y_{s,t} , \tag{4.16}$$

satisfies $\|R^Y\|_{2\alpha} < \infty$. This defines the space of *controlled rough paths*,

$$(Y, Y') \in \mathscr{D}^{2\alpha}_X([0, T], \bar{W}).$$

Although Y' is not, in general, uniquely determined from Y (cf. Remark 4.7 and Section 6 below) we call any such Y' the *Gubinelli derivative* of Y (with respect to X).

Here, $R^Y_{s,t}$ takes values in $\bar{W}$, and the norm $\|\cdot\|_{2\alpha}$ for a function with two arguments is given by (2.3) as before. We endow the space $\mathscr{D}^{2\alpha}_X$ with the semi-norm

$$\|Y, Y'\|_{X,2\alpha} \stackrel{\text{def}}{=} \|Y'\|_\alpha + \|R^Y\|_{2\alpha} . \tag{4.17}$$

As in the case of classical Hölder spaces, $\mathscr{D}^{2\alpha}_X$ is a Banach space under the norm $(Y, Y') \mapsto |Y_0| + |Y'_0| + \|Y, Y'\|_{X,2\alpha}$. This quantity also controls the α-Hölder regularity of Y since, uniformly over X bounded in α-Hölder seminorm,

$$\begin{aligned}\|Y\|_\alpha &\le \|Y'\|_\infty \|X\|_\alpha + T^\alpha \|R^Y\|_{2\alpha} \le |Y'_0| \|X\|_\alpha + T^\alpha \{\|Y'\|_\alpha \|X\|_\alpha + \|R^Y\|_{2\alpha}\} \\ &\le (1 + \|X\|_\alpha)\,(|Y'_0| + T^\alpha \|Y, Y'\|_{X,2\alpha}) \lesssim |Y'_0| + T^\alpha \|Y, Y'\|_{X,2\alpha} . \end{aligned} \tag{4.18}$$

Remark 4.7. Since we only assume that $\|Y\|_\alpha < \infty$, but then impose that $\|R^Y\|_{2\alpha} < \infty$, it is in general the case that a genuine cancellation takes place in (4.16). The question arises to what extent Y determines Y'. Somewhat contrary to the classical situation, where a smooth function has a unique derivative, too much regularity of the underlying rough path $\mathbf{X}$ leads to less information about Y'. For instance, if Y is smooth, or in fact in $\mathcal{C}^{2\alpha}$, and the underlying rough path $\mathbf{X}$ happens to have a path component X that is also $\mathcal{C}^{2\alpha}$, then we may take $Y' = 0$, but as a matter of fact any continuous path Y' would satisfy (4.16) with $\|R\|_{2\alpha} < \infty$. On the other hand, if X is far from smooth, i.e. genuinely rough on all (small) scales, uniformly in all directions, then Y' is uniquely determined by Y, cf. Section 6 below.

Remark 4.8. It is important to note that while the space of rough paths $\mathscr{C}^\alpha$ is not even a vector space, the space $\mathscr{D}^{2\alpha}_X$ is a perfectly normal Banach space for any given $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\alpha$. The twist of course is that the space in question depends in a crucial way on the choice of $\mathbf{X}$. The set of all pairs $(\mathbf{X}; (Y, Y'))$ gives rise to the total space

$$\mathscr{C}^\alpha \ltimes \mathscr{D}^{2\alpha} \stackrel{\text{def}}{=} \bigsqcup_{\mathbf{X}\in\mathscr{C}^\alpha} \{\mathbf{X}\} \times \mathscr{D}_X^{2\alpha},$$

with base space $\mathscr{C}^\alpha$ and "fibres" $\mathscr{D}_X^{2\alpha}$. While this looks reminiscent of fibre-bundles like the tangent bundles of a smooth manifold, it is quite different in the sense that the fibre spaces are in general *not* isomorphic. Loosely speaking, the rougher the underlying path X, the "smaller" is $\mathscr{D}_X^{2\alpha}$, see Chapter 6.

Remark 4.9. While the notion of "controlled rough path" has many appealing features, it does not come with a natural approximation theory. To wit, consider $(X, \mathbb{X}) \in \mathscr{C}_g^\alpha([0,T], \mathbf{R}^d)$ as limit of smooth paths $X_n : [0,T] \to \mathbf{R}^d$ in the sense of Proposition 2.5. Then it is natural to approximate $Y = F(X)$ by the $Y_n = F(X_n)$, which is again smooth (to the extent that F permits). On the other hand, there are no obvious approximations $(Y_n, Y_n') \in \mathscr{D}_{X^n}^{2\alpha}$ for an arbitrary controlled rough path $(Y, Y') \in \mathscr{D}_X^{2\alpha}$.

We are now ready to extend Young's integral to that of a path controlled by X against $\mathbf{X} = (X, \mathbb{X})$. Recall from Lemma 4.1 that $Y = F(X)$, with $Y' = DF(X)$, is somewhat the prototype of a controlled rough path. The definition of the rough integral $\int F(X) d\mathbf{X}$ in terms of compensated Riemann sums, cf. (4.6), then immediately suggests to define the integral of Y against $\mathbf{X}$ by[6]

$$\int_0^1 Y\, d\mathbf{X} \stackrel{\text{def}}{=} \lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} \big(Y_s\, X_{s,t} + Y_s'\, \mathbb{X}_{s,t}\big)\,, \tag{4.19}$$

where we took $\bar{W} = \mathcal{L}(V, W)$ and used the canonical injection $\mathcal{L}(V, \mathcal{L}(V, W)) \hookrightarrow \mathcal{L}(V \otimes V, W)$ in writing $Y_s' \mathbb{X}_{s,t}$. With these notations, the resulting integral takes values in W.

With these notations at hand, it is now straightforward to prove the following result, which is a slight reformulation of [Gub04, Prop.1]:

Theorem 4.10 (Gubinelli). *Let $T > 0$, let $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\alpha([0,T], V)$ for some $\alpha \in \left[\frac{1}{2}, \frac{1}{3}\right)$, and let $(Y, Y') \in \mathscr{D}_X^{2\alpha}\big([0,T], \mathcal{L}(V, W)\big)$. Then there exists a constant C depending only on α such that*

a) The integral defined in (4.19) exists and, for every pair s, t, one has the bound

$$\Big|\int_s^t Y_r\, d\mathbf{X}_r - Y_s X_{s,t} - Y_s'\, \mathbb{X}_{s,t}\Big| \le C\big(\|X\|_\alpha \|R^Y\|_{2\alpha} + \|\mathbb{X}\|_{2\alpha} \|Y'\|_\alpha\big)|t-s|^{3\alpha}\,. \tag{4.20}$$

b) The map from $\mathscr{D}_X^{2\alpha}\big([0,T], \mathcal{L}(V, W)\big)$ to $\mathscr{D}_X^{2\alpha}\big([0,T], W\big)$ given by

[6] Note the abuse of notation: we hide dependence on Y' which in general affects the limit but is usually clear from the context.

$$(Y,Y') \mapsto (Z,Z') := \left(\int_0^{\cdot} Y_t \, d\mathbf{X}_t, Y\right), \tag{4.21}$$

is a continuous linear map between Banach spaces and one has the bound [7]

$$\|Z,Z'\|_{X,2\alpha} \le \|Y\|_\alpha + \|Y'\|_\infty \|\mathbb{X}\|_{2\alpha} + CT^\alpha \left(\|X\|_\alpha \|R^Y\|_{2\alpha} + \|\mathbb{X}\|_{2\alpha}\|Y'\|_\alpha\right) .$$

Proof. Part a) is an immediate consequence of Lemma 4.2, as already pointed out in the proof of Theorem 4.4. The estimate (4.20) was pointed out explicitly in (4.15).

It remains to show the bound on $\|Z,Z'\|_{X,2\alpha}$. Splitting up the left hand side of (4.20) after the first term, using the triangle inequality, gives immediately an α Hölder estimate on $\int_s^t Y_r dX_r = Z_{s,t}$, so that $Z \in \mathcal{C}^\alpha$. ($Z' = Y \in \mathcal{C}^\alpha$ is trivial, by the very nature of Y since it is controlled by X.) Similarly, splitting up the left hand side of (4.20) after the second term, gives a 2α-Hölder type estimate estimate on $\int_s^t Y_r dX_r - Y_s X_{s,t} = Z_{s,t} - Z'_s X_{s,t} =: R^Z_{s,t}$, i.e. on the remainder term in the sense of (4.16). The explicit estimate for $\|Z,Z'\|_{X,2\alpha} = \|Y\|_\alpha + \|R^Z\|_{2\alpha}$ is then obvious. □

Remark 4.11. As in the above theorem, assume that $(X,\mathbb{X}) \in \mathscr{C}^\alpha([0,T],V)$ and consider Y and Z two paths controlled by X. More precisely, we assume $(Y,Y') \in \mathscr{D}^{2\alpha}_X([0,T],\mathcal{L}(\bar{V},W))$ and $(Z,Z') \in \mathscr{D}^{2\alpha}_X([0,T],\bar{V})$, where of course $V,\bar{V},W$ are all Banach spaces. Then, in terms of the abstract integration map $\mathcal{I}$ (cf. the sewing lemma) we may define the integral of Y against Z, with values in W, as follows,

$$\int_s^t Y_u \, dZ_u \overset{\text{def}}{=} (\mathcal{I}\Xi)_{s,t} \,, \quad \Xi_{u,v} = Y_u \, Z_{u,v} + Y'_u Z'_u \mathbb{X}_{u,v} \,. \tag{4.22}$$

Here, we use the fact that $Z'_u \in \mathcal{L}(V,\bar{V})$ can be canonically identified with an operator in $\mathcal{L}(V \otimes V, V \otimes \bar{V})$ by acting only on the second factor, and $Y'_u \in \mathcal{L}(V,\mathcal{L}(\bar{V},W))$ is identified as before with an operator in $\mathcal{L}(V \otimes \bar{V},W)$. The reader may be helped to see this spelled out in coordinates, assuming finite dimensions: using indices i, j in $W,\bar{V}$ respectively, and then k, l in V:

$$(\Xi_{u,v})^i = (Y_u)^i_j (Z_{u,v})^j + (Y'_u)^i_{k,j} (Z'_u)^j_l (\mathbb{X}_{u,v})^{k,l}.$$

Note that, relative to the definition of Ξ in the previous proof, it suffices to replace X by Z and Y' by $Y'Z'$. Making this substitution in $\delta\Xi$, as it appears in the aforementioned proof, then gives

$$\delta\Xi_{s,u,t} = -R^Z_{s,u} X_{u,t} - (Y'Z')_{s,u} \mathbb{X}_{u,t}$$

in the present situation. Clearly $Y'Z' \in \mathcal{C}^\alpha$ and so $\|\delta\Xi\|_\beta$ is finite which allows the proof to go through mutatis mutandis. In particular, (4.20) is valid, with the above substitution, and reads

[7] As in (4.18), this implies $\|Z,Z'\|_{X,2\alpha} \lesssim |Y'_0| + T^\alpha \|Y,Y'\|_{X,2\alpha}$, uniformly over bounded $\mathbf{X}$.

4.4 Stability I: rough integration

Consider $\mathbf{X} = (X, \mathbb{X})$, $\tilde{\mathbf{X}} = (\tilde{X}, \tilde{\mathbb{X}}) \in \mathscr{C}^\alpha$ with $(Y, Y') \in \mathscr{D}_X^{2\alpha}$, $(\tilde{Y}, \tilde{Y}') \in \mathscr{D}_{\tilde{X}}^{2\alpha}$. Although (Y, Y') and $(\tilde{Y}, \tilde{Y}')$ live, in general, in different Banach spaces, the "distance"

$$d_{X,\tilde{X},2\alpha}\Big(Y, Y'; \tilde{Y}, \tilde{Y}'\Big) \stackrel{\text{def}}{=} \|Y' - \tilde{Y}'\|_\alpha + \|R^Y - R^{\tilde{Y}}\|_{2\alpha}$$

will be useful. Even when $X = \tilde{X}$, it is not a proper metric for it fails to separate (Y, Y') and $(Y + cX + \bar{c}, Y' + c)$ for any two constants c and $\bar{c}$. When $X \neq \tilde{X}$, the assertion "zero distance implies $\big(Y, Y'\big) = \big(\tilde{Y}, \tilde{Y}'\big)$" does not even make sense. (The two objects live in completely different spaces!) That said, for every fixed $(X, \mathbb{X}) \in \mathscr{C}^\alpha$, one has (with $R^Y_{s,t} = Y_{s,t} - Y'_s X_{s,t}$ as usual), a canonical map

$$\iota_X : \big(Y, Y'\big) \in \mathcal{C}_X^\alpha \mapsto \big(Y', R^Y\big) \in \mathcal{C}^\alpha \oplus \mathcal{C}_2^{2\alpha} \;.$$

Given $Y_0 = \xi$, this map is injective since one can reconstruct Y by $Y_t = \xi + Y'_0 X_{0,t} + R^Y_{0,t}$. From this point of view, one simply has

$$d_{X,\tilde{X},2\alpha} = \|\iota_X(.) - \iota_{\tilde{X}}(.)\|_{\alpha,2\alpha} \;,$$

and one is back in a normal Banach setting, where $\|\cdot,\cdot\|_{\alpha,2\alpha} = \|\cdot\|_\alpha + \|\cdot\|_{2\alpha}$ is a natural semi-norm on $\mathcal{C}^\alpha \oplus \mathcal{C}_2^{2\alpha}$. (In fact, it is a norm if one only considers elements in $\mathcal{C}^\alpha$ started at 0.) Elementary estimates of the form

$$\big|ab - \tilde{a}\tilde{b}\big| \le \big|a\big|\,\big|b - \tilde{b}\big| + \big|a - \tilde{a}\big|\,\big|\tilde{b}\big| \tag{4.26}$$

then lead to, with a constant $C = C_R$,

$$\begin{aligned}
\big|Y_{s,t} - \tilde{Y}_{s,t}\big| &= \Big|\big(Y'_{0,s} - Y'_0\big)X_{s,t} + \big(\tilde{Y}'_{0,s} + \tilde{Y}'_0\big)\tilde{X}_{s,t} + R^Y_{s,t} - R^{\tilde{Y}}_{s,t}\Big| \\
&\le C|t-s|^\alpha\Big(\big|Y'_0 - \tilde{Y}'_0\big| + \big\|X - \tilde{X}\big\|_\alpha + \big\|Y'_{0,\cdot} - \tilde{Y}'_{0,\cdot}\big\|_\infty + \big\|R^Y - R^{\tilde{Y}}\big\|_\alpha\Big) \\
&\le C|t-s|^\alpha\Big(\big|Y'_0 - \tilde{Y}'_0\big| + \big\|X - \tilde{X}\big\|_\alpha + T^\alpha\Big(\big\|Y' - \tilde{Y}'\big\|_\alpha + \big\|R^Y - R^{\tilde{Y}}\big\|_{2\alpha}\Big)\Big) \;,
\end{aligned}$$

provided $|Y'_0|$, $\|Y'\|_\infty$, $\|X\|_\alpha$, and also with tilde, are bounded by R. It follows that

$$\big\|Y - \tilde{Y}\big\|_\alpha \le C\Big(\big\|X - \tilde{X}\big\|_\alpha + \big|Y'_0 - \tilde{Y}'_0\big| + T^\alpha d_{X,\tilde{X},2\alpha}\big(Y, Y'; \tilde{Y}, \tilde{Y}'\big)\Big) \;. \tag{4.27}$$

An estimate of the proper α-Hölder norm of $Y - \tilde{Y}$ (rather than its semi-norm) is obtained by adding $\big|Y_0 - \tilde{Y}_0\big|$ to both sides.

Theorem 4.16 (Stability of rough integration). *For $\alpha \in \big(\frac{1}{3}, \frac{1}{2}\big]$ as before, consider $\mathbf{X} = \big(X, \mathbb{X}\big)$, $\tilde{\mathbf{X}} = \big(\tilde{X}, \tilde{\mathbb{X}}\big) \in \mathscr{C}^\alpha$, $\big(Y, Y'\big) \in \mathscr{D}_X^{2\alpha}$, $\big(\tilde{Y}, \tilde{Y}'\big) \in \mathscr{D}_{\tilde{X}}^{2\alpha}$ in a bounded set, in the sense*

$$|Y'_0| + \|Y, Y'\|_{X,2\alpha} \le M, \quad \varrho_\alpha(0, \mathbf{X}) \equiv ||X||_\alpha + \|\mathbb{X}\|_{2\alpha} \le M,$$

with identical bounds for $(\tilde{X}, \tilde{\mathbb{X}})$, $(\tilde{Y}, \tilde{Y}')$, *for some* $M < \infty$. *Define*

$$(Z, Z') := \left(\int_0^{\cdot} Y \, d\mathbf{X}, Y \right) \in \mathscr{D}_X^{2\alpha} ,$$

and similarly for $(\tilde{Z}, \tilde{Z}')$. *Then, the following local Lipschitz estimates holds true,*

$$d_{X,\tilde{X},2\alpha}(Z, Z'; \tilde{Z}, \tilde{Z}') \le C\Big(\varrho_\alpha(\mathbf{X}, \tilde{\mathbf{X}}) + |Y_0' - \tilde{Y}_0'| + T^\alpha d_{X,\tilde{X},2\alpha}(Y, Y'; \tilde{Y}, \tilde{Y}')\Big), \tag{4.28}$$

and also

$$\|Z - \tilde{Z}\|_\alpha \le C\Big(\varrho_\alpha(\mathbf{X}, \tilde{\mathbf{X}}) + |Y_0 - \tilde{Y}_0| + |Y_0' - \tilde{Y}_0'| + T^\alpha d_{X,\tilde{X},2\alpha}(Y, Y'; \tilde{Y}, \tilde{Y}')\Big) , \tag{4.29}$$

where $C = C_M = C(M, \alpha)$ *is a suitable constant.*

Proof. (The reader is advised to review the proofs of Theorems 4.4, 4.10.) We first note that (4.27) applied to $Z, \tilde{Z}$ (note: $Z_0' - \tilde{Z}_0 = Y_0 - \tilde{Y}$) shows that (4.29) is an immediate consequence of the first estimate (4.28). Thus, we only need to discuss the first estimate. By definition of $d_{X,\tilde{X},2\alpha}$, we need to estimate

$$\|Z' - \tilde{Z}'\|_\alpha + \|R^Z - R^{\tilde{Z}}\|_{2\alpha} = \|Y - \tilde{Y}\|_\alpha + \|R^Z - R^{\tilde{Z}}\|_{2\alpha}.$$

Thanks to (4.27), the first summand is clearly bounded by the right-hand side of (4.28). For the second summand we recall

$$R^Z_{s,t} = Z_{s,t} - Z_s' X_{s,t} = \int_s^t Y \, d\mathbf{X} - Y_s X_{s,t} = (\mathcal{I}\Xi)_{s,t} - \Xi_{s,t} + Y_s' \mathbb{X}_{s,t}$$

where $\Xi_{s,t} = Y_s X_{s,t} + Y_s' \mathbb{X}_{s,t}$ and similar for $R^{\tilde{Z}}$. Setting $\Delta = \Xi - \tilde{\Xi}$, we use (4.11) with $\beta = 3\alpha$ and Ξ replaced by Δ, so that

$$\begin{aligned} \big|R^Z_{s,t} - R^{\tilde{Z}}_{s,t}\big| &= \big|(\mathcal{I}\Delta)_{s,t} - \Delta_{s,t}\big| + \big|Y_s'\mathbb{X}_{s,t} - \tilde{Y}_s'\tilde{\mathbb{X}}_{s,t}\big| \\ &\le C\|\delta\Delta\|_{3\alpha}|t-s|^{3\alpha} + \big|Y_s'\mathbb{X}_{s,t} - \tilde{Y}_s'\tilde{\mathbb{X}}_{s,t}\big| , \end{aligned}$$

where $\delta\Delta_{s,u,t} = R^{\tilde{Y}}_{s,u}\tilde{X}_{u,t} - R^Y_{s,u}X_{u,t} + \tilde{Y}'_{s,u}\tilde{\mathbb{X}}_{u,t} - Y'_{s,u}\mathbb{X}_{u,t}$. We then conclude with some elementary estimates of the type (4.26), just like in the proof of Theorem 4.10.
□

4.5 Controlled rough paths of lower regularity

Recall that we showed in Section 2.3 how an α-Hölder rough path $\mathbf{X}$ could be defined as a path with values in the p-step nilpotent Lie group $G^{(p)}(\mathbf{R}^d) \subset T^{(p)}(\mathbf{R}^d)$, with $p = \lfloor 1/\alpha \rfloor$. It does not seem obvious at all a priori how one would define a controlled

$$\left\|\mathcal{M}_{T_0}(Y,Y') - \mathcal{M}_{T_0}(\tilde{Y},\tilde{Y}')\right\|_{X,2\alpha} \le \frac{1}{2}\|Y-\tilde{Y},Y'-\tilde{Y}'\|_{X,2\alpha}$$

and so $\mathcal{M}_{T_0}(\cdot)$ admits a unique fixed point $(Y,Y') \in \mathcal{B}_{T_0}$, which is then the unique solution Y to (8.1) on the (possibly rather small) interval $[0,T_0]$. Noting that the choice of T_0 can again be done uniformly in the starting point, the solution on $[0,1]$ is then constructed iteratively as before. □

In many situations, one is interested in solutions to an equation of the type

$$dY = f_0(Y,t)\,dt + f(Y,t)\,d\mathbf{X}_t\,, \tag{8.11}$$

instead of (8.6). On the one hand, it is possible to recast (8.11) in the form (8.6) by writing it as an RDE for $\hat{Y}_t = (Y_t,t)$ driven by $\hat{\mathbf{X}}_t = (\hat{X},\hat{\mathbb{X}})$ where $\hat{X} = (X_t,t)$ and $\hat{\mathbb{X}}$ is given by $\mathbb{X}$ and the "remaining cross integrals" of X_t and t, given by usual Riemann-Stieltjes integration. However, it is possible to exploit the structure of (8.11) to obtain somewhat better bounds on the solutions. See [FV10b, Ch. 12].

8.6 Stability III: Continuity of the Itô–Lyons map

We now obtain continuity of solutions to rough differential equations as function of their (rough) driving signals.

Theorem 8.5 (Rough path stability of the Itô–Lyons map). *Let $f \in \mathcal{C}_b^3$ and, for $\alpha \in \left(\frac{1}{3},\frac{1}{2}\right]$, let $(Y,f(Y)) \in \mathscr{D}_X^{2\alpha}$ be the unique RDE solution given by Theorem 8.4 to*

$$dY = f(Y)\,d\mathbf{X}, \quad Y_0 = \xi \in W\,.$$

Similarly, let $(\tilde{Y},f(\tilde{Y}))$ be the RDE solution driven by $\tilde{\mathbf{X}}$ and started at $\tilde{\xi}$ where $\mathbf{X},\tilde{\mathbf{X}} \in \mathscr{C}^\alpha$. Assuming

$$\|\mathbf{X}\|_\alpha, \|\tilde{\mathbf{X}}\|_\alpha \le M < \infty$$

we have the local Lipschitz estimates

$$d_{X,\tilde{X},2\alpha}(Y,f(Y);\tilde{Y},f(\tilde{Y})) \le C_M\Big(|\xi-\tilde{\xi}| + \varrho_\alpha(\mathbf{X},\tilde{\mathbf{X}})\Big)\,,$$

and also

$$\|Y-\tilde{Y}\|_\alpha \le C_M\Big(|\xi-\tilde{\xi}| + \varrho_\alpha(\mathbf{X},\tilde{\mathbf{X}})\Big)\,,$$

where $C_M = C(M,\alpha,f)$ is a suitable constant.

Remark 8.6. The proof only uses the (apriori) information that RDE solutions remain bounded if the driving rough paths do, combined with basic stability properties of rough integration and composition.

Proof. Recall that, for given $\mathbf{X} \in \mathscr{C}^\alpha$, the RDE solution $(Y,f(Y)) \in \mathscr{D}_X^{2\alpha}$ is constructed as the unique fixed point of

$$\mathcal{M}_T(Y, Y') := (Z, Z') := \left(\xi + \int_0^{\cdot} f(Y_s) d\mathbf{X}_s, f(Y_\cdot)\right) \in \mathscr{D}_X^{2\alpha} ,$$

and similarly for $\tilde{\mathcal{M}}_T(\tilde{Y}, f(\tilde{Y})) \in \mathcal{C}_{\tilde{X}}^{\alpha}$. Then, thanks to the fixed point property

$$(Y, f(Y)) = (Y, Y') = (Z, Z') = (Z, f(Y)) ,$$

(similarly with tilde) and the local Lipschitz estimate for rough integration, Theorem 4.16, and writing $(\Xi, \Xi') := \big(f(Y), f(Y)'\big)$ for the integrand, we obtain the bound

$$\begin{aligned} d_{X,\tilde{X},2\alpha}(Y, Y'; \tilde{Y}, \tilde{Y}') &= d_{X,\tilde{X},2\alpha}(Z, Z'; \tilde{Z}, \tilde{Z}') \\ &\lesssim \varrho_\alpha(\mathbf{X}, \tilde{\mathbf{X}}) + |\xi - \tilde{\xi}| + T^\alpha d_{X,\tilde{X},2\alpha}(\Xi, \Xi'; \tilde{\Xi}, \tilde{\Xi}') , \end{aligned}$$

Thanks to the local Lipschitz estimate for composition, Theorem 7.5, uniform in $T \le 1$,

$$d_{X,\tilde{X},2\alpha}(\Xi, \Xi'; \tilde{\Xi}, \tilde{\Xi}') \lesssim \varrho_\alpha(\mathbf{X}, \tilde{\mathbf{X}}) + |\xi - \tilde{\xi}| + d_{X,\tilde{X},2\alpha}(Y, f(Y); \tilde{Y}, f(\tilde{Y})) .$$

In summary, for some constant $C = C(\alpha, f, M)$, we have the bound

$$\begin{aligned} d_{X,\tilde{X},2\alpha}(Y, f(Y); \tilde{Y}, f(\tilde{Y})) \le C\big(&\varrho_\alpha(\mathbf{X}, \tilde{\mathbf{X}}) + |\xi - \tilde{\xi}| \\ &+ T^\alpha d_{X,\tilde{X},2\alpha}(Y, f(Y); \tilde{Y}, f(\tilde{Y}))\big) . \end{aligned}$$

By taking $T = T_0(M, \alpha, f)$ smaller, if necessary, we may assume that $CT^\alpha \le 1/2$, from which it follows that

$$d_{X,\tilde{X},2\alpha}(Y, f(Y); \tilde{Y}, f(\tilde{Y})) \le 2C\big(\varrho_\alpha(\mathbf{X}, \tilde{\mathbf{X}}) + |\xi - \tilde{\xi}|\big) ,$$

which is precisely the required bound. The bound on $\|Y - \tilde{Y}\|_\alpha$ then follows as in (4.29), and these bounds can be iterated to cover a time interval of arbitrary (fixed) length. □

8.7 Davie's definition and numerical schemes

Fix $f \in \mathcal{C}_b^2(W, \mathcal{L}(V, W))$ and $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\beta([0,T], V)$ with $\beta > \frac{1}{3}$. Under these assumptions, the rough differential equation $dY = f(Y) d\mathbf{X}$ makes sense as well-defined integral equation. (In Theorem 8.4 we used additional regularity, namely $\mathcal{C}_b^3$, to establish existence of a *unique* solution on $[0,T]$.) By the very definition of an RDE solution, unique or not, $(Y, f(Y)) \in \mathscr{D}_X^{2\beta}$ i.e.

$$Y_{s,t} = f(Y_s) X_{s,t} + \mathrm{O}\big(|t-s|^{2\beta}\big)$$