Ergodic Properties of Markov Processes

Exercises for week 8

Exercise 1 Consider the movement of a king on a chessboard who, at every timestep, chooses one of the possible 8 (or less if he is on the border) moves independently and with equal probabilities. Show that the position of the king forms a Markov process which is reversible with respect to its invariant measure and give an expression for the invariant measure $\pi$.

Exercise 2 Let $M$ be a subset of the set of probability measures on $\mathbb{R}^n$. Show that a sufficient condition for $M$ to be tight is that there exists a function $F: \mathbb{R}^n \to \mathbb{R}_+$ with $\lim_{R \to \infty} \inf \{ F(x) \mid \|x\| \geq R \} = \infty$ and a constant $C$ such that

$$\int_{\mathbb{R}^n} F(x) \mu(dx) < C,$$

for every $\mu \in M$.

Exercise 3 Show that a sequence of delta-measures $\{\delta_{x_n}\}_{n \geq 0}$ on a complete separable metric space $\mathcal{X}$ converges weakly to a delta measure $\delta_x$ if and only if the sequence $\{x_n\}$ converges to $x$. Similarly, show that the sequence is tight if and only if the closure of the set $\{x_n \mid n \geq 0\}$ is compact. Note: consider first the case $\mathcal{X} = \mathbb{R}$.

Exercise 4 Use the central limit theorem to show that the sequence $\{\mu_n\}$ of measures on $\mathbb{Z}$ given by the laws of the simple random walk at time $n$ is not tight.

Exercise 5 Let $\mathcal{H}$ be a Hilbert space with an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ and define a sequence of measures $\mu_n$ by

$$\mu_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{e_k}.$$

Show that this sequence of measures is not tight in $\mathcal{H}$. Let $\mathcal{H}'$ be the Hilbert space obtained by completing $\mathcal{H}$ under the norm

$$\|x\|^2 = \sum_{n=1}^{\infty} \frac{\langle e_n, x \rangle}{n}.$$

Show that the sequence of measures $\mu_n$ viewed as measures on $\mathcal{H}'$ is tight and actually converges weakly to a limit. Give this limit.

Exercise 6 Let $\{\xi_n\}$ be a sequence of i.i.d. random variables with values in the space of continuous functions $\mathcal{C}([0,1], \mathbb{R})$ and such that $\mathbb{E} \sup_{t \in [0,1]} |\xi_n(t)|^2 < \infty$. Let $x_n$ be the real-valued Markov process defined so that given $x_n$, $x_{n+1}$ is the solution at time 1 to the differential equation

$$\frac{dx(t)}{dt} = x(t) - x^3(t) + \xi_n(t), \quad x(0) = x_n.$$

Show that this Markov process admits an invariant probability measure.

Hint: Show that there exists a constant $C$ such that $\mathbb{E}(x_{n+1}^2 \mid x_n) \leq C + \frac{1}{2} x_n^2$. And conclude that the law of $x_n$ generates a tight family of probability measures on $\mathbb{R}$.
**Exercise 7 (Wiener measure)** Let $\mathcal{X}$ be the space of continuous functions from $[0,1]$ into $\mathbb{R}$ and let $\{\xi_n\}_{n \geq 0}$ be a sequence of i.i.d. $\mathcal{N}(0,1)$ random variables. Let $\varphi: \mathbb{R} \to [0,1]$ be the function defined by $\varphi(t) = \max\{\min\{t,1\},0\}$ and define a sequence $\{x_n\}$ of $\mathcal{X}$-valued random variables by

$$x_n(t) = \frac{N-1}{\sqrt{N}} \sum_{n=0}^{N-1} \xi_n \varphi(t - \frac{n}{N}).$$

Show that the sequence $\mu_n$ of measures on $\mathcal{X}$ given by the laws of $x_n$ is tight so that there exists a probability measure $W$ on $\mathcal{X}$ and a subsequence $n_k$ such that $\mu_{n_k} \Rightarrow W$ weakly. Show that one has actually $\mu_n \Rightarrow W$ weakly by showing that the law of $(x_{t_1}, \ldots, x_{t_k})$ under $\mu_n$ converges to a limiting distribution as $n \to \infty$.

**Hint:** Remember that the Arzela-Ascoli theorem states that a set $A \subset \mathcal{X}$ is relatively compact if and only if it is bounded and the functions in $A$ are equicontinuous.