Ergodic Properties of Markov Processes

Exercises for week 2

**Exercise 1** Show that if $\mathcal{F}'$ is the trivial $\sigma$-algebra, i.e. $\mathcal{F}' = \{\phi, \Omega\}$, then $X' = E(X \mid \mathcal{F}')$ is constant and equal to the expectation of $X$.

**Exercise 2** Show that continuous functions are Borel-measurable. Give an example of a Borel-measurable function from $\mathbb{R}$ to $\mathbb{R}$ which is not continuous.

**Exercise 3** Let $\Omega = [0, 1]^2$, $P(dx, dy) = (x + y)\, dx\, dy$, and let $(X, Y)$ be a pair of random variables defined by $X(x, y) = x$ and $Y(x, y) = y$. Let $\mathcal{F}_Y$ be the $\sigma$-algebra generated by $Y$. Find an explicit expression for $E(X \mid \mathcal{F}_Y)$ and give a function $f$ such that $E(X \mid \mathcal{F}_Y) = f \circ Y$.

**Exercise 4** Show that $\mathcal{F}_1 \vee \mathcal{F}_2$ can equivalently be characterised by the expressions:

- $\mathcal{F}_1 \vee \mathcal{F}_2 = \sigma\{A \cup B \mid A \in \mathcal{F}_1 \text{ and } B \in \mathcal{F}_2\}$,
- $\mathcal{F}_1 \vee \mathcal{F}_2 = \sigma\{A \cap B \mid A \in \mathcal{F}_1 \text{ and } B \in \mathcal{F}_2\}$,

where $\sigma\mathcal{G}$ denotes the smallest $\sigma$-algebra containing $\mathcal{G}$.

**Exercise 5** Let $\Omega = \{1, \ldots, 6\}^3$. We interpret elements of $\Omega$ as the possible outcomes of throwing a dice three times. Describe the $\sigma$-algebra $\mathcal{F}$ corresponding to knowing the value of the largest of the three throws.

* **Exercise 6** Show the following elementary properties of conditional expectations:
  - If $\mathcal{F}_1 \subset \mathcal{F}_2$, then $E(E(X \mid \mathcal{F}_2) \mid \mathcal{F}_1) = E(E(X \mid \mathcal{F}_1) \mid \mathcal{F}_2) = E(X \mid \mathcal{F}_1)$.
  - Find an example that shows that in general $E(E(X \mid \mathcal{F}_2) \mid \mathcal{F}_1) \neq E(E(X \mid \mathcal{F}_1) \mid \mathcal{F}_2)$.
  - If $Y$ is $\mathcal{F}_1$-measurable, then $E(Y \mid \mathcal{F}_1) = Y \, E(X \mid \mathcal{F}_1)$.
  - If $\mathcal{F}_1 \subset \mathcal{F}_2$, and $Y$ is $\mathcal{F}_2$-measurable then $E(Y \, E(X \mid \mathcal{F}_2) \mid \mathcal{F}_1) = E(Y \, E(X \mid \mathcal{F}_1))$.

  **Hint** For the first part, use the fact that if $\mathcal{F}_1 \subset \mathcal{F}_2$, then any $\mathcal{F}_1$-measurable function is also $\mathcal{F}_2$-measurable.

**Exercise 7** You have probably seen Lebesgue measurable functions defined through the property that $f^{-1}(A)$ is Lebesgue measurable for every open set $A$. Show that every Borel measurable function is also Lebesgue measurable but that the converse is not true in the case of functions from $\mathbb{R}$ to $\mathbb{R}$.

Show that if $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ are Borel measurable functions, then $g \circ f$ is also Borel measurable. This property is not true for Lebesgue measurable functions. Try to find a continuous function $f : \mathbb{R} \to \mathbb{R}$ and a Lebesgue measurable function $g$ (you can take an indicator function for $g$) such that $g \circ f$ is not Lebesgue measurable.

**Hint:** Remember that every measurable set $A$ of positive Lebesgue measure contains a subset $A' \subset A$ which is not Lebesgue measurable. (Take this statement for granted if you haven’t seen it before.) Another useful ingredient for the construction of $f$ is the Cantor function $D$ (also called Devil’s staircase), depicted here. Use the fact that if $C$ is the Cantor set, then $D(C)$ is a set of Lebesgue measure 1.