Ex: $T = \operatorname{span}\{\mathbb{1}, \mathbb{W}_i, \mathbb{W}_{ij} : i,j = 1 \dots m\}$

$$\begin{array}{llll} \text{deg:} & 0 & \alpha & 2\alpha \qquad\qquad d = 1 \\ & & \underset{(1/3,\,1/2)}{\cap} & \end{array}$$

Fix smooth $W : \mathbb{R} \to \mathbb{R}^m$

$$\Pi_s \mathbb{1} = 1 \qquad (\Pi_s \mathbb{W}_i)(t) = \delta W^i_{s,t} := W^i_t - W^i_s$$

$$(\Pi_s \mathbb{W}_{ij})(t) = \int_s^t \delta W^i_{s,r} \, dW^j(r)$$

Want : $\Gamma_{s,r} \in G$ s.t. $\Pi_s \cdot \Gamma_{s,r} = \Pi_r$

$\Rightarrow \quad \Gamma_{s,r} \mathbb{1} = \mathbb{1}$

$$\Gamma_{s,r} \mathbb{W}_i = \mathbb{W}_i + \delta W^i_{r,s} \cdot \mathbb{1}$$

$$\Gamma_{s,r} \mathbb{W}_{ij} = \mathbb{W}_{ij} + \delta W^i_{r,s} \cdot \mathbb{W}_j + \int_r^s (\dots) \cdot \mathbb{1}$$

$$\int_s^t \delta W^i_{s,s'} \, dW^j(s') + \int_r^s \delta W^i_{r,r'} \, dW^j(r') + \delta W^i_{r,s} \, \delta W^j_{s,t} = \int_r^t (\ \ )$$

$\gamma \in (\alpha, 2\alpha)$ $\qquad F \in \mathcal{D}^\gamma \iff |F(s) - \Gamma_{s,t} F(t)|_\alpha \lesssim |t-s|^{\gamma-\alpha}$

$$F(t) = f(t)\,\mathbb{1} + f_i(t) \cdot \mathbb{W}_i$$

$$|f_i(t) - f_i(s)| \lesssim |t-s|^{\gamma-\alpha}$$

$$|f(s) - f(t) - \delta W^i_{t,s}\, f_i(t)| \lesssim |t-s|^\gamma .$$

One can define an 'integral' $F \mapsto \int_0 F(s)\, dW^i(s)$

$$"\int_0 F(s)\,dW^j(s)" = \lim_{|\mathcal{P}|\to 0} \sum_{[s,t]\in\mathcal{P}} \underbrace{\Big(f(s)\,\delta W^j_{s,t} + f^i(s)\big(\Pi_s \mathbb{W}_{ij}\big)(t) \Big)}_{A_{s,t}}$$

Claim (Sewing lemma; Gubinelli)

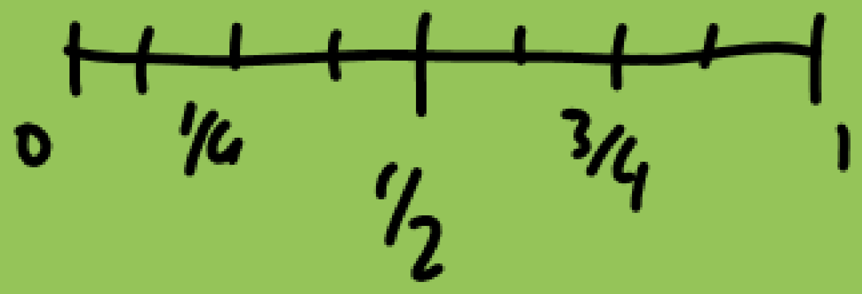
If $\exists\ \beta > 1$ s.t. $|A_{s,t} - A_{s,u} - A_{u,t}| \lesssim |t-s|^\beta$

$\alpha > 0$ $\quad |A_{s,t}| \lesssim |t-s|^\alpha$

$\Rightarrow \exists!$ fcn $\bar{A}$ s.t. $\quad |\delta\bar{A}_{s,t} - A_{s,t}| \lesssim |t-s|^\beta$

"Proof" Go along dyadic partitions $\mathcal{P}^{(n)}$

$\uparrow$ mesh 2^{-n}



$$A^{(n)} = \sum_{[s,t]\in\mathcal{P}^{(n)}} A_{s,t}$$

$u = \frac{s+t}{2}$

$$|A^{(n+1)} - A^{(n)}| = \sum_{[s,t]\in\mathcal{P}^{(n)}} \underbrace{\left(A_{s,t} - A_{s,u} - A_{u,t}\right)}_{|t-s|^{\beta} = 2^{-\beta n}} \le 2^{(1-\beta)n}$$

$\uparrow$ 2^n terms

<u>Reconstruction thm</u>: $\gamma > 0$ $\exists!$ $\mathcal{R} : \mathcal{D}^\gamma \to \mathcal{D}'$

s.t. $|(\mathcal{R}F - \Pi_x F(x))(\varphi_x^\lambda)| \lesssim \lambda^\gamma$

Choose $\rho : \mathbb{R}^d \to \mathbb{R}$ $\mathcal{C}_0^\infty$ s.t. $\rho(-x) = \rho(x)$

$$\int \rho(x) \cdot x^\ell \, dx = \delta_{0,\ell} \quad \forall \; |\ell| \leq |\alpha|$$

$$\rho^{(n)}(x) = 2^{nd} \cdot \rho(2^n x) \qquad \rho^{(n,m)} = \rho^{(n)} * \rho^{(n+1)} * \dots * \rho^{(m)}$$

$$\varphi^{(n)} = \lim_{m \to \infty} \rho^{(n,m)}$$

Lemma ($\alpha<0$): Assume $\zeta_n : \mathbb{R}^d \to \mathbb{R}$ s.t. $|\zeta_n| \lesssim 2^{-\alpha n}$

and $\zeta_n = \rho^{(n)} * \zeta_{n+1}$. Then $\lim\limits_{n\to\infty} \zeta_n = \zeta$

exists in $\mathcal{C}^\beta$ $\forall \beta<\alpha$ and $\zeta_n = \varphi^{(n)} * \zeta$.

If for some $x \in \mathbb{R}^d$ $\gamma > \alpha$ $\quad |\zeta_n(y)| \lesssim 2^{-\alpha n}\left(|x-y|^{\gamma-\alpha}\right.$

then $\quad |\zeta(\psi_x^\lambda)| \lesssim \lambda^\gamma$. $\quad \left.(|y-x|\le 1) + 2^{-(\gamma-\alpha)n}\right)$

generic test fcn.

Proof: Wanted: $\left|\psi^\lambda * (\zeta_n - \zeta_{n+1})\right| \lesssim \lambda^\beta 2^{-(\alpha-\beta)n}$

If $\lambda \le 2^{-n} \Rightarrow$ follows from $(\_) \le 2^{\alpha n}$

$\lambda \geq 2^{-n}$ $\quad |\ldots| = \left|(\psi^\lambda * \rho^{(n)} - \psi^\lambda) * \zeta_{n+1}\right|$

$\left|(\psi^\lambda * \rho^{(n)} - \psi^\lambda)(x)\right| = \left|\left(\rho^{(n)} * (\psi^\lambda - T_x^{(N)})\right)(x)\right|$

Taylor expansion of ψ^λ around x

$\lesssim 2^{-nN} \lambda^{-N-d} \Rightarrow \int |\psi^\lambda * \rho^{(n)} - \psi^\lambda| \lesssim 2^{-nN} \lambda^{-N}$

$$\Rightarrow \quad |\ldots| \lesssim 2^{-\alpha n} \cdot 2^{-nN} \cdot \lambda^{-N} \leq \lambda^{\beta} 2^{-n(\alpha-\beta)}$$

$$\zeta(\psi_x^\lambda) = \underbrace{\zeta_n(\psi_x^\lambda)}_{\text{easy}} + \underbrace{\sum_{k \geq n} (\zeta_{k+1} - \zeta_k)(\psi_x^\lambda)}_{\text{Same strategy as before.}}$$

$$\lambda \in [2^{-(n+1)}, 2^{-n}]$$

Idea of proof: Morally, $\mathcal{R}F(x) = \left(\prod_x F(x)\right)(x)$

$$\left(\mathcal{R}^{(n,n)} F\right)(x) = \left(\prod_x F(x)\right)\left(\varphi_x^{(n)}\right)$$

$$\mathcal{R}^{(n,m)} F = \rho^{(n,m-1)} * \mathcal{R}^{(m,m)} F$$