$$
\begin{tikzcd}
\mathfrak{F} \arrow[d, leftrightarrow] & \times & \mathcal{M} \arrow[loop above, "g"] \arrow[rr] & & \mathcal{D}^{n} \arrow[d] \\
\mathcal{F} \arrow[loop below, "g"] & \times & \zeta_{\varepsilon} \arrow[u] \arrow[rr] & & \text{Dist/Fcn}
\end{tikzcd}
$$

<u>Def</u>: A regularity structure consists of:

- a graded vector space

$$T = \bigoplus_{\alpha \in A} T_\alpha \qquad A \subset \mathbb{R} \text{ discrete bdd below}$$

each T_α a Banach space

- a group $G \subset L(T,T)$ s.t. $\forall \Gamma \in G$

$$\forall \tau \in T_\alpha \qquad \Gamma\tau - \tau \in \bigoplus_{\beta < \alpha} T_\beta = T_{<\alpha}$$

Morphisms

$$\Phi : (T, G) \longrightarrow (T', G')$$

consists of

$\varphi : T \to T'$ s.t. $\varphi : T_\alpha \to T'_\alpha$

$\theta : G' \to G$ s.t

$$\Gamma' \varphi(\tau) = \varphi(\theta(\Gamma')\tau)$$

$\Phi = (\varphi, \theta)$

$\Phi' = (\varphi', \theta')$ $\Rightarrow$ $\Phi \circ \Phi' := (\varphi \circ \varphi', \theta' \circ \theta)$

$$(\varphi' \circ \varphi)\big((\theta \circ \theta')(\Gamma'')\,\tau\big) = \varphi'\Big(\varphi\big(\theta(\theta'(\Gamma''))\tau\big)\Big)$$

$$= \varphi'\big(\theta'(\Gamma'')\,\varphi(\tau)\big) = \Gamma''\,\varphi'(\varphi(\tau))$$

$$= \Gamma''(\varphi' \circ \varphi)(\tau)$$

The space of models. Fix some $\mathbb{R}^d$ with some scaling

Fix a reg. struc. (T, G). A <u>model</u> (Π, Γ) for (T, G) consists of

$$\mathbb{R}^d \ni x \longmapsto \Pi_x \in L(T, \mathcal{D}'(\mathbb{R}^d))$$

$$(x,y) \longmapsto \Gamma_{xy} \in G \quad \text{s.t.} \quad \Pi_x \Gamma_{xy} = \Pi_y$$

$$\Gamma_{xy} \cdot \Gamma_{yz} = \Gamma_{xz}$$

We also enforce the following analytic bounds:

$$\forall \tau \in T_\alpha \qquad |(\Pi_x \tau)(\varphi_x^\lambda)| \leq C |\tau|_\alpha \lambda^\alpha$$

$$|\Gamma_{xy}\tau|_\beta \leq C |x-y|^{\alpha-\beta} |\tau|_\alpha$$

uniformly over $\lambda \in (0,1]$, $\varphi \in \mathcal{B}_r$, loc. uniform in x.

$r > -\min A$

The spaces $\mathcal{D}^\gamma$. Fix reg. struc (T, G) and model (Π, Γ).

$F \in \mathcal{D}^\gamma$ if $F : \mathbb{R}^d \to T_{<\gamma}$ s.t

$$|F(x) - \Gamma_{xy} F(y)|_\alpha \leq C |x-y|^{\gamma-\alpha}$$

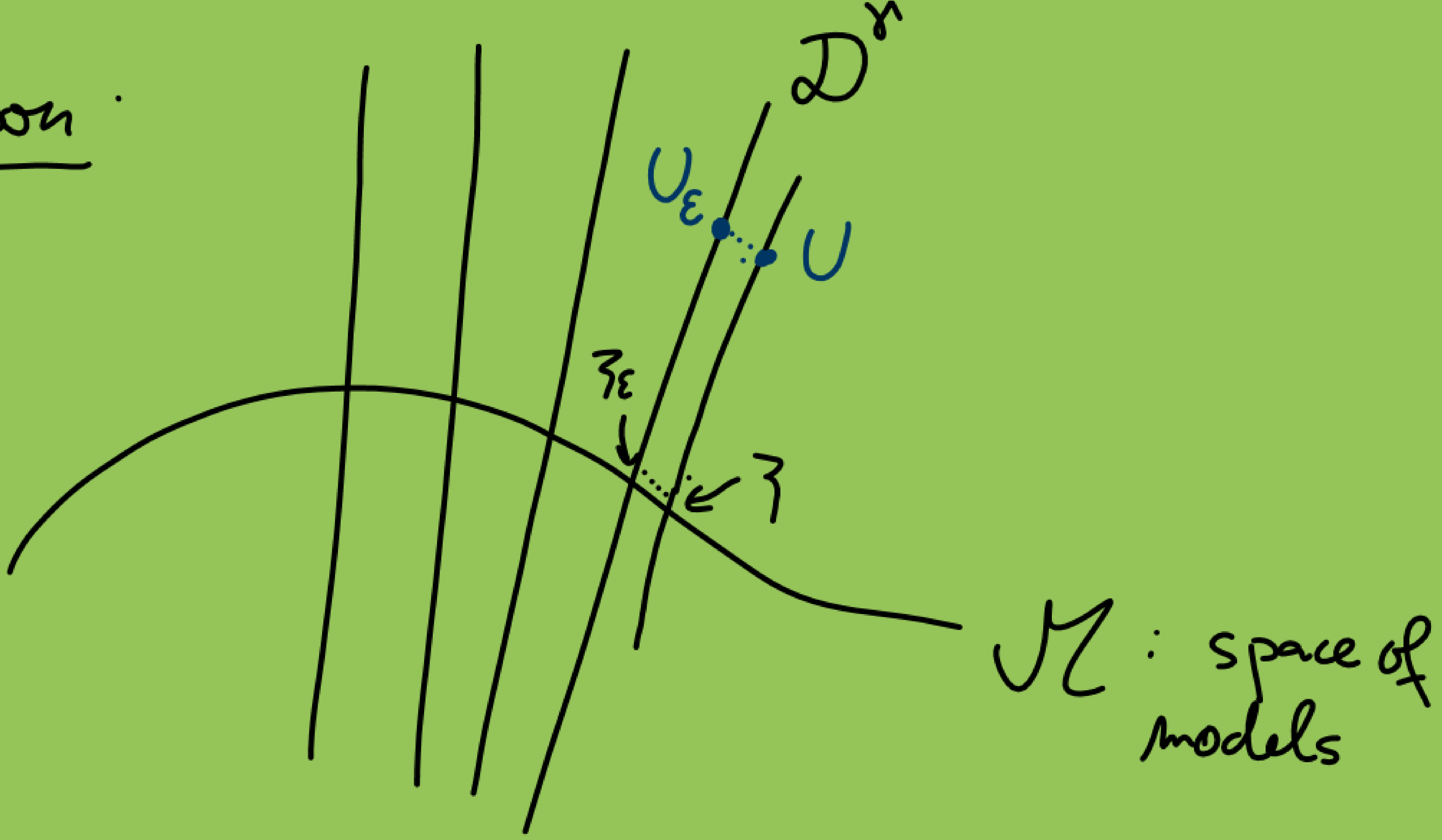
Thm: $\forall \gamma > 0 \quad \exists! \, \mathcal{R} : \mathcal{D}^\gamma \longrightarrow \mathcal{D}'(\mathbb{R}^d)$

s.t. $\quad \left| \left(\mathcal{R}F - \Pi_x F(x) \right)(\varphi_x^\lambda) \right| \leq c \, \|F\|_{\mathcal{D}^\gamma} \cdot \lambda^\gamma$

$\forall \gamma < 0 \quad \exists \, \mathcal{R} : \mathcal{D}^\gamma \to \mathcal{D}'(\mathbb{R}^d)$ s.t. $\Uparrow$

Rem: $\mathcal{R}F \in \mathcal{C}^{\min A}$

Cartoon:



$\mathcal{D}^\gamma$
U_ε
U
ξ_ε
ξ
$\mathcal{M}$: space of models

Ex: Controlled rough paths (T. Lyons, M. Gubinelli)

$$dx_i = F_{ij}(x)\, dW_j$$

$$\begin{aligned} \Rightarrow \quad x_i(t) \simeq {} & x_i(s) 1 + F_{ij}(x(s)) \cdot \delta W_j(s,t) \\ & + \partial_k F_{ij}(x(s))\, F_{k\ell}(x(s)) \int_s^t \delta W_\ell(s,r)\, dW_j(r) \end{aligned}$$

$$T = \mathbb{R} \oplus \mathbb{R}^{m} \oplus \mathbb{R}^{m^2}$$

$$\begin{array}{ccc}
\deg 0 & \deg \alpha & \deg 2\alpha \\
\rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\
\mathbb{1} & \mathbb{W}_i & \mathbb{W}_{ij}
\end{array}$$

$$(\Pi_s \mathbb{1})(t) = 1$$

$$(\Pi_s \mathbb{W}_{ij})(t) = \int_s^t \delta W_i(s,r)\, dW_j(r)$$

$$(\Pi_s \mathbb{W}_i)(t) = \delta W_i(s,t)$$