$$F_i \in \mathcal{D}^{\gamma_i}_{\alpha_i} \Rightarrow F_1 \cdot F_2 \in \mathcal{D}^{\gamma} \qquad \gamma = (\gamma_1 + \alpha_2) \wedge (\gamma_2 + \alpha_1)$$

No guarantee that $\mathcal{R}(F_1 \cdot F_2) = \mathcal{R}F_1 \cdot \mathcal{R}F_2$??

Assume V sector of degree 0 s.t. $V_0 = \text{span}\{\mathbb{1}\}$

where $\mathbb{1} \cdot \tau = \tau \quad \forall \tau \in V$. (Product $\cdot$ on V)

Assume $\cdot$ is associative & commutative

For $U \in \mathcal{D}^\gamma(V)$, write $U_x = u_x \mathbb{1} + \tilde{u}_x$

with $\tilde{u} \in T_{\geq \beta}$ some $\beta > 0$. Note, if

$\gamma \geq \beta$, then $u \in \mathcal{C}^\beta$:

$$|u_x - u_y - \underbrace{(\Gamma_{xy} \tilde{u}_y)_0}_{\lesssim |x-y|^\beta}| \lesssim |x-y|^\gamma$$

$F \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$

Projection on $T_{<\gamma}$ w.r.t. $\bullet$ in V

<u>Def</u>: $F(U)_x := Q_{<\gamma} \sum_{k} \frac{F^{(k)}(u_x)}{k!} \tilde{u}_x^{k}$

<u>Prop</u>: $U \in \mathcal{D}^\gamma \Rightarrow F(U) \in \mathcal{D}^\gamma$

<u>Proof</u> : $\quad F(U)_x - \Gamma_{xy} F(U)_y$

$$= \sum_{k} \frac{F^{(k)}(u_x)}{k!} \tilde{u}_x^{\,k} - \sum_{k} \frac{F^{(k)}(u_y)}{k!} \left(\Gamma_{xy} \tilde{u}_y\right)^{k}$$

$$= \sum_{k,\ell} \frac{F^{(k+\ell)}(u_y)}{k!\,\ell!} (u_x - u_y)^{\ell} \tilde{u}_x^{\,k} - (\ldots)$$

$$= \sum_{m} \frac{F^{(m)}(u_y)}{m!} \left[\left((u_x - u_y)\mathbf{1} + \tilde{u}_x \right)^{m} - \left(\Gamma_{xy} \tilde{u}_y \right)^{m} \right]$$

$$= \sum_m \frac{F^{(m)}(u_y)}{m!} \underbrace{\left((u_x - u_y)\mathbb{1} + \tilde{u}_x - \Gamma_{xy}\tilde{u}_y\right)}_{U(x) - \Gamma_{xy} U(y)} \cdot \underbrace{(\ldots\ldots)}_{\deg \geq 0}$$

$$\Rightarrow |(\sim)|_\alpha \lesssim \sum_{\beta < \alpha} |U(x) - \Gamma_{xy} U(y)|_\beta \lesssim |x-y|^{\gamma - \alpha}$$

$\square$

Derivatives "in space"

Sector V, linear maps $\mathscr{D}_i : V \to T$ s.t.

- $\tau \in V_\alpha \Rightarrow \mathscr{D}_i \tau \in T_{\alpha-1}$
- $\Gamma \in G$, $\tau \in V \Rightarrow \Gamma \mathscr{D}_i \tau = \mathscr{D}_i \Gamma \tau$

Prop: $F \in \mathscr{D}^\gamma(V) \Rightarrow \mathscr{D}_i F \in \mathscr{D}^{\gamma-1}$ $\square$

Prop: If model (Π, Γ) is such that

$$\Pi_x \mathcal{D}_i \tau = \partial_i \Pi_x \tau \qquad \forall \tau \in V$$

Then $\mathcal{R} \mathcal{D}_i F = \partial_i \mathcal{R} F \qquad F \in \mathcal{D}^\gamma(V)$

$\gamma > 1$

Proof:

$$\Big| \Big(\underbrace{\mathcal{R} \mathcal{D}_i F - \Pi_x \mathcal{D}_i F(x)}_{\partial_i \mathcal{R} F - \partial_i \Pi_x F(x)} \Big)(\varphi_x^\lambda) \Big| \leq \lambda^{\gamma - 1}$$

$\square$

$$\partial_t u = \Delta u + F(u, \nabla u, \xi, \dots)$$

$$\Updownarrow$$

$$u(t) = P_t u_0 + \int_0^t P_{t-s} \cdot F(u_s, \nabla u_s, \dots) \, ds$$

$$= P_t u_0 + P * \mathbb{1}_{t>0} \cdot F(\dots)$$

Convolution with singular kernel.

$$P(t,x) = \frac{1}{t^{d/2}} e^{-\frac{|x|^2}{t}} \in \mathcal{C}^\infty(\mathbb{R}^{d+1} \setminus \{0\})$$

$$= \frac{1}{\Big(\underbrace{(t^2+x^4)^{1/4}}_{\|t,x\|}\Big)^d} \cdot \underbrace{\left(1+\left(\frac{x^2}{t}\right)^2\right)^{d/4} e^{-\frac{|x|^2}{t}}}_{G\left(\frac{x^2}{t}\right)}$$

One has $$\left|\partial_t^{k_0}\cdot\partial_x^{k_1} P\right| \lesssim \frac{1}{\|t,x\|^{d+\underbrace{|k_1|+2k_0}_{|k|}}}$$

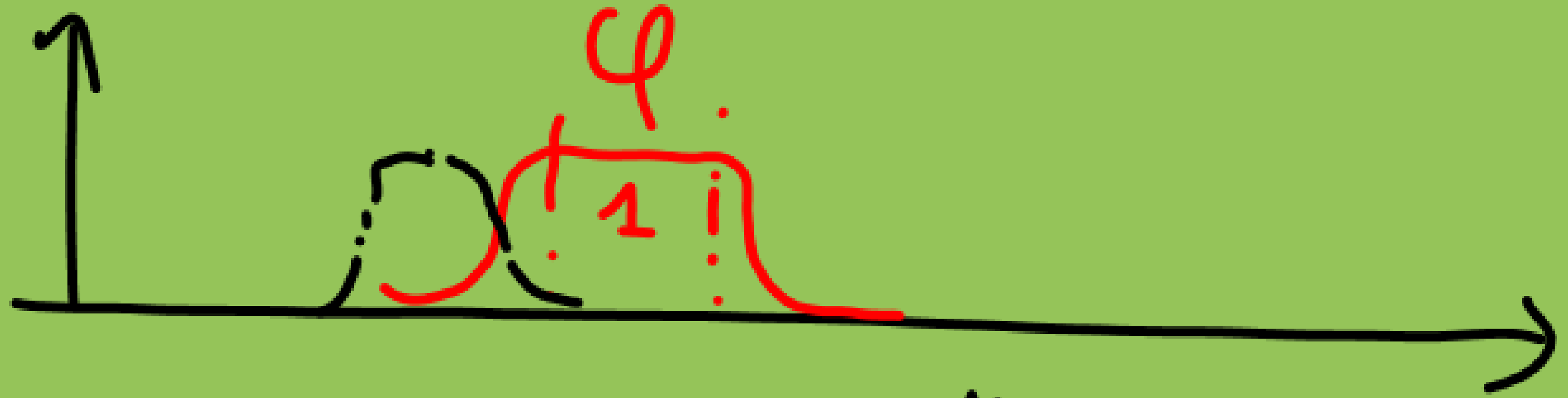
<u>Def</u>: $K : \mathbb{R}^d\setminus\{0\}\to\mathbb{R}$ is β-regularising if

- K compactly supported (say in $B(0,1)$)
- $|D^k K(x)| \lesssim |x|^{\beta-d-|k|}$

Lemma: If K is β-regularising, then one can write $K = \sum_{n \geq 0} K_n$ with

$K_n \in C\, 2^{-\beta n}\, \mathcal{B}_r^{2^{-n}}$ (any fixed $r > 0$)

Proof:



$\sum_{n \geq 0} \varphi(2^n x) = 1$

$K_n(x) = K(x) \cdot \varphi(2^n |x|)$

Always assume $\bar{T} \subset T$, with $\bar{T}$ "Taylor polynomials".

$$\Gamma_{xy} X^k = (X + x - y)^k$$

<u>Def</u>: Given sector V, an "integration operator" of order β is linear map $\mathcal{I} : V \to T$ s.t.

- $\tau \in V_\alpha \implies \mathcal{I}\tau \in T_{\alpha+\beta}$
- $\Gamma \in G, \tau \in V \implies \Gamma \mathcal{I}\tau - \mathcal{I}\Gamma\tau \in \bar{T}$

<u>Wanted:</u> Operator $\mathcal{K} : \mathcal{D}^{\gamma} \longrightarrow \mathcal{D}^{\gamma+\beta}$

s.t.

- $\mathcal{R}\mathcal{K}F = K * \mathcal{R}F$
- $(\mathcal{K}F)(x) - \mathcal{I}F(x) \in \mathcal{T}$

<u>Claim:</u> This is possible for "admissible" models

Def: A model (Π, Γ) is admissible if

$$\Pi_x \mathcal{I}\tau = K * \Pi_x \tau - \sum_n \sum_{|\ell| < \deg \mathcal{I}\tau} \frac{(\cdot - x)^\ell}{\ell!} \underbrace{(\Pi_x \tau)(D^{(\ell)} K_n(\cdot - x))}_{2^{-n(\underbrace{\deg\tau + \beta}_{\deg \mathcal{I}\tau} - |\ell|)}}$$