$$\rho^{(n,m)} = \rho^{(n)} * \cdots * \rho^{(m)} \qquad \varphi^{(n)} = \rho^{(n,\infty)}$$

$$f \in \mathcal{D}^{\gamma} \quad \gamma > 0 \quad \Longrightarrow \quad |(\mathcal{R}f - \Pi_x f(x))(\varphi_x^{\lambda})| \lesssim \lambda^{\gamma}$$

$$(\mathcal{R}^{(n,n)} f)(y) := (\Pi_y f(y))(\varphi_y^{(n)}) \lesssim 2^{-\underline{\alpha} n}$$

$$\mathcal{R}^{(n,m)} f := \rho^{(m,m-1)} * \mathcal{R}^{(m,m)}$$

(Otto – Weber)

$$\left(\mathcal{R}^{(n,m)} - \mathcal{R}^{(n,m+1)}\right) f(x) = \int \rho_x^{(n,m-1)}(y)$$

$$\int \rho_y^{(m)}(z) \Big[\underbrace{\left(\Pi_y f(y)\right)\left(\varphi_z^{(m+1)}\right) - \left(\Pi_z f(z)\right)\left(\varphi_z^{(m+1)}\right)} \Big] \, dz \, dy$$

$$\left(\Pi_y \big(\underbrace{f(y) - \Gamma_{yz} f(z)} \big)\right)\left(\varphi_z^{(m+1)}\right)$$

$$\deg \alpha : \lesssim |y-z|^{\gamma-\alpha} \, 2^{-m\alpha} \lesssim 2^{-m\gamma}$$

$\Rightarrow$ For fixed n, $\mathcal{R}^{(n,m)} f$ is Cauchy in $\mathcal{C}$ as $m \to \infty$. Limit $\mathcal{R}^{(n)} f$

Since $\mathcal{R}^{(n,m)} f = \rho^{(n)} * \mathcal{R}^{(n+1,m)} f$,

$$\Rightarrow \mathcal{R}^{(n)} f = \rho^{(n)} * \mathcal{R}^{(n+1)} f$$

We also have $|(\mathcal{R}^{(n)} f)| \lesssim 2^{-\alpha n}$

By last week's result, $\mathcal{R}^{(n)} f$ is Cauchy in $\underline{\mathcal{C}^{\alpha} \ \forall \alpha < \underline{\alpha}}$, limit $\mathcal{R} f \in \mathcal{C}^{\underline{\alpha}}$.

Set $f_x(y) := \Gamma_{yx} f(x) = \Gamma_{yz} \cdot \Gamma_{zx} f(x)$

$= \Gamma_{yz} f_x(z)$

$\Pi_y f_x(y) = \Pi_x f(x) \ \forall y$

$\Longrightarrow \Pi_x f(x) = \mathcal{R} f_x$

<u>Wanted:</u> $\left|\mathcal{R}^{(n)}(f-f_x)(y)\right| \lesssim 2^{-\underline{\alpha} n}\left(|x-y|^{\gamma-\bar{\alpha}} + 2^{-(\gamma-\bar{\alpha})n}\right)$

$$\left|\mathcal{R}^{(n,n)}(f-f_x)(y)\right| = \left|\left(\Pi_y\left(f(y)-\Gamma_{yx}f(x)\right)\right)\left(\varphi_y^{(n)}\right)\right|$$

$$\lesssim \sum_{\alpha<\gamma} |y-x|^{\gamma-\alpha}\, 2^{-n(\alpha-\underline{\alpha})} \cdot 2^{-n\underline{\alpha}}$$

$$\lesssim 2^{-n\underline{\alpha}}\left(|y-x|^{\gamma-\underline{\alpha}} + 2^{-n(\gamma-\underline{\alpha})}\right) \qquad \square$$

Consequences: $\mathcal{C}^\alpha \ni f \times \zeta \in \mathcal{C}^\beta$ $\quad \alpha > 0$, $\beta < 0$

$$\downarrow$$

$f \cdot \zeta \in \mathcal{C}^\beta$ Continuous iff $\alpha + \beta > 0$

Proof:

$$T = \mathrm{span}\{ \underbrace{X^k}_{\deg = |k|}, \underbrace{\Xi X^k}_{\deg = \beta + |k|} \}$$

$G = (\mathbb{R}^d, +)$

$$G \ni h \cdot X^k = (X - h)^k$$

$$h \cdot \Xi X^k = \Xi (X - h)^k$$

Model:

$$(\Pi_x^{\xi} X^{k})(y) = (y-x)^{k}$$

$$\xi \to \Pi^{\xi}$$

$$(\Pi_x^{\xi} \Xi X^{k})(y) = \xi(y) \cdot (y-x)^{k}$$

$$f \in \mathcal{C}^{\alpha} \Longrightarrow F \in \mathcal{D}^{\alpha} \qquad F(x) = \sum_{|k|<\alpha} \frac{f^{(k)}(x)}{k!} X^{k}$$

$$\Longrightarrow \Xi F \in \mathcal{D}^{\alpha+\beta}$$

$$\Longrightarrow \xi \cdot f := \underbrace{\mathcal{R}(\Xi F)}_{\in \mathcal{C}^{\beta}}$$

Example: $T = \langle \mathbb{1}, S, S^2 \rangle \qquad K > 0$

deg $\quad 0 \quad -K \quad -2K$

$G = id$

$$\Pi_x^{(n)} \mathbb{1} = 1 \qquad (\Pi_x^{(n)} S^2)(y) = \left(\sin(ny)\, n^{K/2}\right)^2 - \frac{n^K}{2}$$

$$f \in \mathcal{D}^\gamma = f_0(x) \cdot \mathbb{1} + f_1(x) \cdot S \implies \begin{array}{l} f_0 \in \mathcal{C}^\gamma \\ f_1 \in \mathcal{C}^{\gamma + K} \end{array}$$

$$(\mathcal{R}^{(n)} f)(x) = f_0(x) + f_1(x) \sin(nx)$$

$$\mathcal{R}^{(n)} f^2 = (\mathcal{R}^{(n)} f)^2$$

$$\left| \left(\Pi^{(n)} S \right) \left(\varphi_x^\lambda \right) \right| \leq \begin{cases} 1 \\ \frac{1}{\lambda n} \text{ (int. by parts)} \end{cases} \leq \frac{1}{\lambda^K \cdot n^K}$$

$$\Rightarrow \Pi^{(n)} \Rightarrow \Pi \qquad \Pi \mathbb{1} = 1 \quad \Pi S = 0 \quad \Pi S^2 = 1/2$$

$$\Rightarrow (f_0 + f_1 \sin nx)^2 \to f_0^2 + \frac{1}{2} f_1^2 \quad \text{in } \mathcal{C}^{-2K}$$

Products:

<u>Def</u>: $T = \bigoplus_\alpha T_\alpha$ Given $V = \bigoplus_\alpha V_\alpha$

$V_\alpha \subset T_\alpha$ linear

V is a "sector" if $\Gamma\tau \in V$ $\forall \tau \in V$ $\Gamma \in G$

V has "regularity" α if $V \subset T_{\geq \alpha}$

Def: A bilinear map $* : V \times W \to T$ for sectors V, W is a 'product' if

- $\deg(\tau * \sigma) = \deg \tau + \deg \sigma$
- $\Gamma(\tau * \sigma) = \Gamma\tau * \Gamma\sigma \quad \forall\, \Gamma \in G$

Define $\mathcal{D}^{\gamma}_{\alpha} = \{ F \in \mathcal{D}^{\gamma} : F(x) \in T_{\geq \alpha} \} \quad (\alpha \leq 0)$

$F_1 \in \mathcal{D}^{\gamma_1}_{\alpha_1}(V) \quad F_2 \in \mathcal{D}^{\gamma_2}_{\alpha_2}(W) \Rightarrow F_1 * F_2 \in \mathcal{D}^{\gamma}_{\alpha}$

for $\alpha = \alpha_1 + \alpha_2$

$\gamma = (\gamma_1 + \alpha_2) \wedge (\gamma_2 + \alpha_1)$

$$
\begin{aligned}
&\left| F_1(x)F_2(x) - \Gamma_{xy}F_1(y)\cdot\Gamma_{xy}F_2(y) \right|_\beta \\
&\leq \left| \big(F_1(x) - \Gamma_{xy}F_1(y)\big)F_2(x) \right|_\beta + \left| F_1(x)\big(F_2(x) - \Gamma_{xy}F_2(y)\big) \right|_\beta \\
&\quad + \left| \big(F_1(x) - \Gamma_{xy}F_1(y)\big)\big(F_2(x) - \Gamma_{xy}F_2(y)\big) \right|_\beta \\
&\leq |x-y|^{\gamma_1-\beta+\alpha_2} + |x-y|^{\gamma_2-\beta+\alpha_1} + |x-y|^{\gamma_1+\gamma_2-\beta} \qquad \square
\end{aligned}
$$