$$\mathcal{I} : V \to T \supset \bar{T}$$

$$\Gamma \mathcal{I} - \mathcal{I} \Gamma \in L(V, \bar{T})$$

<u>Admissible</u>: $\quad \Pi_x X^k = (\cdot - x)^k$

degree $\mathcal{I}\tau$

$$\Pi_x \mathcal{I}\tau = K * \Pi_x \tau - \text{Taylor}_x$$

$$\Rightarrow \exists\, \mathcal{K} : \mathcal{D}^\gamma(V) \to \mathcal{D}^{\gamma+\beta} \quad (\gamma + \beta \notin \mathbb{N})$$

$\beta > 0$: $\mathcal{Z}^\beta = \{ \eta : |\eta(\varphi_x^\lambda)| \lesssim \lambda^\beta \quad \varphi \in \mathcal{B}_r$

s.t. $\int P \cdot \varphi = 0 \quad \deg P \leq \beta \}$

<u>Thm</u>: K β-regularising $\Rightarrow$ $\eta \in \mathcal{Z}^\alpha$

implies $K * \eta \in \mathcal{Z}^{\alpha+\beta}$

Prop: $\varphi \in \mathcal{B}_n^{\lambda} \quad \psi \in \mathcal{B}_n^{\mu} \Longrightarrow \varphi * \psi \in \mathcal{B}_n^{\lambda+\mu}$

If $\lambda \leq \mu$ & $\int P \cdot \varphi = 0 \quad \deg P < \gamma \in n$

$$\Longrightarrow \quad \varphi * \psi \in C \left(\frac{\lambda}{\mu}\right)^{\gamma} \mathcal{B}_{\lfloor n-\gamma \rfloor}^{\lambda+\mu}$$

Proof:

$$(\varphi * \psi)(x) = \int \varphi(x-y)\psi(y)\,dy$$

$$= \int \varphi(x-y)\Big(\psi(y) - \sum_{|\xi|<\gamma} \frac{D^{(\xi)}\psi(x)}{\xi!}(y-x)^{\xi}\Big)\,dy$$

$$|\cdot - | \leq \int \varphi(x-y) \cdot \frac{|y-x|^\gamma}{\mu^\gamma} \, dy$$

$$\lesssim \left(\frac{\lambda}{\mu}\right)^\gamma$$

$\square$

<u>Proof of Schauder</u> $\quad \eta \in \mathcal{Z}^\alpha \qquad K_n \in 2^{-\beta n} \mathcal{B}^{2^{-n}}_r$

$$(K * \eta)(\varphi_x^\lambda) = \sum_{n \geq 0} \eta(K_n * \varphi_x^\lambda)$$

$\underline{2^{-n} \le \lambda}$: $K_n * \varphi_x^\lambda \in 2^{-\beta n} \mathcal{B}_r^{2\lambda}$

$\Rightarrow |\ldots| \lesssim 2^{-\beta n} \cdot \lambda^\alpha \Rightarrow \sum |\ldots| \lesssim \lambda^{\alpha+\beta}$

$\underline{2^{-n} \ge \lambda}$: $K_n * \varphi_x^\lambda \in 2^{-\beta n} \left(\frac{\lambda}{2^{-n}}\right)^{\alpha+\beta+\delta} \mathcal{B}^{2\cdot 2^{-n}}_{\lfloor r-(\alpha+\beta)\rfloor + \delta}$ (some $\delta > 0$)

$|\ldots| \lesssim 2^{-\beta n} \left(\frac{\lambda}{2^{-n}}\right)^{\alpha+\beta+\delta} \cdot (2^{-n})^\alpha \lesssim \lambda^{\alpha+\beta+\delta} 2^{n\delta}$

$\Rightarrow \sum |\ldots| \lesssim \lambda^{\alpha+\beta+\delta} \lambda^{-\delta} = \lambda^{\alpha+\beta}$

$\square$

Extension of reg. structure

$(T, G) \qquad V \subset T \qquad \hat{T} = T \oplus \hat{V}$

Define $\hat{V}_{\alpha+\beta} \cong V_\alpha \qquad \mathcal{I}\underset{\in \hat{V}}{(\tau, 0)} = \underbrace{(0, \tau)}_{\mathcal{I}\tau}$

$$\hat{G} = G \times \bigoplus_{\alpha} \hat{V}_{\alpha+\beta} \otimes \bar{T}_{<\alpha+\beta}$$

assume $\notin \mathbb{N}$

$\hat{\Gamma} \in \hat{G} \qquad \hat{\Gamma} = (\Gamma, \{T^\tau\}_{\tau \in V})$

Fixed basis

Same fixed basis

$$\hat{\Gamma}_1 \sum \tau = \sum \Gamma_1 \tau + T_1^{\tau}$$

$$\hat{\Gamma}_2 \hat{\Gamma}_1 \sum \tau = \sum \Gamma_2 \Gamma_1 \tau + T_2^{\Gamma_1 \tau} + \Gamma_2 T_1^{\tau}$$

$$\Rightarrow \hat{\Gamma}_2 \cdot \hat{\Gamma}_1 = \left(\Gamma_2 \cdot \Gamma_1 , \left\{ T_2^{\Gamma_1 \tau} + \Gamma_2 T_1^{\tau} \right\}_{\tau} \right)$$

Wanted

$$|(\Pi_x \mathcal{I}\tau)(\varphi_x^\lambda)| \lesssim \lambda^{\overset{\deg\tau}{\alpha+\beta}}$$

$$(\Pi_x \mathcal{I}\tau)(\varphi_x^\lambda) = \sum_{n \geq 0} (\Pi_x \tau)(K^\lambda_{n,x})$$

$$K^\lambda_{n,x}(z) = \int \varphi_x^\lambda(y)\Big(K_n(y-z) - \sum_{|\ell|<\alpha+\beta} \frac{D^{(\ell)}K_n(x-z)}{\ell!}\cdot(y-x)^\ell\Big)\,dy$$

$\underline{2^{-n} \leq \lambda}$:

$$|(\Pi_x \tau)(\ldots)| \lesssim 2^{-\beta n} \lambda^{\alpha} + \sum_{|e| < \alpha+\beta} \lambda^{|e|} \cdot 2^{-n(\alpha+\beta-|e|)}$$

$$\left|\sum (\ldots)\right| \lesssim \lambda^{\alpha+\beta}$$

$\underline{2^{-n} \geq \lambda}$: Test fcn at scale $\sim 2^{-n}$ of size

$$2^{-\beta n} \cdot \lambda^{\alpha+\beta+\delta} \quad 2^{n(\alpha+\beta+\delta)}$$

$\square$

Multilevel Schauder

with values in V

Want: Construct $\mathcal{K} : \mathcal{D}^{\gamma} \to \mathcal{D}^{\gamma+\beta}$ $(\gamma+\beta \notin \mathbb{N})$

s.t. $\mathcal{R}\mathcal{K}F = K * \mathcal{R}F$

Step 1: Consider $\hat{T} \supset T$ given by $\hat{T} = T \oplus \mathrm{span}\{\mathbb{F}\}$ $\quad \deg \mathbb{F} = \gamma$

$$\hat{G} = G \times V_{<\gamma} \qquad (\Gamma, H)\mathbb{F} = \mathbb{F} + H$$

Extend the model by

$$\Pi_x \mathbb{F} = \mathcal{R}F - \Pi_x F(x)$$

$$\Gamma_{xy} \mathbb{F} = \mathbb{F} + F(x) - \Gamma_{xy} F(y)$$

Extend again to get new symbol $\mathcal{I}\mathbb{F}$ & model

$$\Pi_x \mathcal{I}\mathbb{F} = K * \Pi_x \mathbb{F} - \mathrm{Taylor}_x$$

Consider $\hat{F}(x) = \mathbb{F} + F(x)$

Then $\Gamma_{xy}\hat{F}(y) = \mathbb{F} + F(x) - \Gamma_{xy}F(y) + \Gamma_{xy}F(y)$

$$= \hat{F}(x)$$

In particular $\Pi_x \hat{F}(x) = \mathcal{R}F \quad \forall x.$

$\Pi_x \mathcal{I}\hat{F}(x) = K * \mathcal{R}F - \mathrm{Taylor}_x$

$$\hat{\mathcal{K}} F(x) := \mathcal{I}\hat{F}(x) + \mathrm{Taylor}_x$$

Then, as before $\hat{\mathcal{K}}F(x) = \Gamma_{xy}\hat{\mathcal{K}}F(y)$

$$\Rightarrow \quad \hat{\mathcal{K}}F \in \mathcal{D}^{\infty} \quad \& \quad \mathcal{R}\hat{\mathcal{K}}F = K * \mathcal{R}F$$

$$\hat{\mathcal{K}}F = \underset{\deg \gamma+\beta}{\underline{\mathcal{I}\mathbb{F}}} + \underset{\substack{\cap \\ T_{<\gamma+\beta}}}{\mathcal{K}F} \quad \Rightarrow \quad \mathcal{K}F \in \mathcal{D}^{\gamma+\beta}$$

$\square$